SHOCK WAVES AND COMPACTONS FOR FIFTH-ORDER NONLINEAR DISPERSION EQUATIONS

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Abstract. The following question is posed: to justify that the *standing shock wave*

$$S_-(x) = -\operatorname{sign} x = -\begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases}$$

is a correct "entropy" solution of fifth-order nonlinear dispersion equations (NDEs),

$$u_t = -(uu_x)_{xxxx} \quad \text{and} \quad u_t = -(uu_{xxxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+.$$

These two quasilinear degenerate PDEs are chosen as typical representatives, so other similar $(2m+1)$th-order NDEs with no divergence structure admit such shocks.

As a related second problem, the opposite shock $S_+(x) = -S_-(x) = \operatorname{sign} x$ is shown to be a non-entropy solution that gives rise to a continuous *rarefaction wave* for $t > 0$.

Formation of shocks is also studied for the fifth-order in time NDE

$$u_{ttttt} = (uu_x)_{xxxx}.$$

On the other hand, related NDEs are shown to admit smooth *compactons*, e.g., for

$$u_t = -(|u|u_x)_{xxxx} + |u|u_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

which are of changing sign. Nonnegative ones are nonexistent in general (not robust).

1. Introduction: nonlinear dispersion PDEs, Riemann's Problems, and main directions of study

1.1. Three main problems: entropy shocks and rarefaction waves for fifth-order NDEs. Let us introduce our basic models that are five fifth-order nonlinear dispersion equations (NDEs). These are ordered by numbers of derivatives inside and outside the quadratic differential operators involved:

$$u_t = -uu_{xxxxx} \qquad \big(\text{NDE–}(5,0)\big), \tag{1.1}$$

$$u_t = -(uu_{xxxx})_x \qquad \big(\text{NDE–}(4,1)\big), \tag{1.2}$$

$$u_t = -(uu_{xxx})_{xx} \qquad \big(\text{NDE–}(3,2)\big), \tag{1.3}$$

$$u_t = -(uu_{xx})_{xxx} \qquad \big(\text{NDE–}(2,3)\big), \tag{1.4}$$

$$u_t = -(uu_x)_{xxxx} \qquad \big(\text{NDE–}(1,4)\big). \tag{1.5}$$

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The only fully divergent operator is in the last NDE–$(1,4)$ that, being written as

$$u_t = -(uu_x)_{xxxx} \equiv -\tfrac{1}{2}\,(u^2)_{xxxxx} \quad \big(\text{NDE–}(1,4) = \text{NDE–}(0,5)\big), \tag{1.6}$$

becomes also the NDE–$(0,5)$, or simply the NDE–5. This completes the list of such quasilinear degenerate PDEs under consideration.

Before explaining the physical significance of such models and their role in general PDE theory, we pose those three main problems for the above NDEs:

(I) *Problem "Blow-up to S_-"* (Section 2): *to show that the shock of the shape* $-\operatorname{sign} x$ *can be obtained by blow-up limit from regular (at least, continuous) solutions* $u_-(x,t)$ *of (1.1)–(1.5) in* $\mathbb{R} \times (0,T)$*, so that*

$$u_-(x,t) \to S_-(x) = -\operatorname{sign} x = \begin{cases} 1 & \textit{for } x<0, \\ -1 & \textit{for } x>0, \end{cases} \quad \textit{as} \quad t \to T^- \quad \textit{in} \quad L^1_{\rm loc}(\mathbb{R}). \tag{1.7}$$

(II) *Riemann's Problem S_+ (RP+)* (Section 3): *to show that the shock*

$$S_+(x) = \operatorname{sign} x = \begin{cases} -1 & \textit{for } x<0, \\ 1 & \textit{for } x>0, \end{cases} \tag{1.8}$$

for NDEs (1.1)–(1.5) generates "rarefaction waves" that are continuous for $t>0$.

(III) *Riemann's Problem S_- (RP$-$)* (Section 4): *to show that*

$$S_-(x) \ \textit{ is an "entropy" shock wave, and } S_+(x) \ \textit{ is not.} \tag{1.9}$$

(IV) *Problem: nonuniqueness/entropy* (Section 5): *to show that a single point "gradient catastrophe" for the NDE (1.5) leads to a shock wave, which is principally nonunique.*

In Section 6, we discuss first these problems in application to various NDEs including the following rather unusual one:

$$u_{ttttt} = (uu_x)_{xxxx}, \tag{1.10}$$

which indeed can be reduced to a first-order system that, nevertheless, is not hyperbolic. The main convenient mathematical feature of (1.10) is that it is in the *normal form*, so it obeys the Cauchy–Kovalevskaya theorem that guarantees local existence of a unique analytic solution (this adds extra flavour to our δ-entropy test). Regardless this, (1.10) is shown to create shocks in finite time and rarefaction waves for other discontinuous data.

In Section 7, as the last "opposite to shocks problem" and as a typical example, we consider the following perturbed version of the NDE (1.5):

$$u_t = -(|u|u_x)_{xxxx} + |u|u_x, \tag{1.11}$$

which is written for solutions of changing sign by replacing u^2 by the monotone function $|u|u$. This is essential, and we show that (1.11) admits compactly supported travelling wave (TW) solutions of changing sign called *compactons*. More standard in literature *nonnegative compactons* of fifth-order NDEs such as (1.11) and others are shown to be nonexistent in general in the sense that these are not *robust*, i.e., do not exhibit continuous dependence on parameters of PDEs and/or small perturbations of nonlinearities.

1.2. **A link to classic entropy shocks for conservation laws.** The above problems **(I)**–**(III)** are classic for entropy theory of 1D conservation laws from the 1950s. Shock waves first appeared in gas dynamics that led to mathematical theory of entropy solutions of the first-order conservation laws such as *Euler's equation*

$$u_t + uu_x = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+. \tag{1.12}$$

Entropy theory for PDEs such as (1.12) was created by Oleinik [30, 31] and Kruzhkov [26] (equations in $\mathbb{R}^N$) in the 1950–60s; see details on the history, main results, and modern developments in the well-known monographs [1, 9, 42]. Note that first analysis of the formation of shocks was performed by Riemann in 1858 [34]; see the history in [3].

According to entropy theory for conservation laws such as (1.12), it is well-known that (1.9) holds. This means that

$$u_-(x,t) \equiv S_-(x) = -\operatorname{sign} x \tag{1.13}$$

is the unique entropy solution of the PDE (1.12) with the same initial data $S_-(x)$. On the contrary, taking S_+-type initial data yields the continuous *rarefaction wave* with a simple similarity piece-wise linear structure,

$$u_0(x) = S_+(x) = \operatorname{sign} x \quad \Longrightarrow \quad u_+(x,t) = g(\tfrac{x}{t}) = \begin{cases} -1 & \text{for } \; x < -t, \\ \frac{x}{t} & \text{for } \; |x| < t, \\ 1 & \text{for } \; x > t. \end{cases} \tag{1.14}$$

Our goal is to justify the same conclusions for fifth-order NDEs, where of course the rarefaction wave in the RP+ is supposed to be different from that in (1.14).

We now return to main applications of the NDEs.

1.3. **NDEs from theory of integrable PDEs and water waves.** Talking about odd-order PDEs under consideration one should mention that these naturally appear in classic theory of integrable PDEs, with those representatives as the *KdV equation*,

$$u_t + uu_x = u_{xxx}, \tag{1.15}$$

the *fifth-order KdV equation*,

$$u_t + u_{xxxxx} + 30\, u^2 u_x + 20\, u_x u_{xx} + 10\, u u_{xxx} = 0,$$

and others from shallow water theory. These are *semilinear* dispersion equations, which being endowed with smooth semigroups, generate smooth flows, so discontinuous weak solutions are unlikely; see references in [20, Ch. 4].

The situation is changed for the quasilinear case. In particular, for the quasilinear *Harry Dym equation*

$$u_t = u^3 u_{xxx}\,, \tag{1.16}$$

which is one of the most exotic integrable soliton equations; see [20, § 4.7] for survey and references therein. Here, (1.16) indeed belongs to the NDE family, though it seems that semigroups of its discontinuous solutions were never under scrutiny. In addition, integrable

equation theory produced various hierarchies of quasilinear higher-order NDEs, such as the fifth-order *Kawamoto equation* [25]

$$u_t = u^5 u_{xxxxx} + 5\,u^4 u_x u_{xxxx} + 10\,u^5 u_{xx} u_{xxx}. \tag{1.17}$$

We can extend this list talking about possible quasilinear extensions of the integrable *Lax's seventh-order KdV equation*

$$u_t + [35u^4 + 70(u^2 u_{xx} + u(u_x)^2) + 7(2uu_{xxxx} + 3(u_{xx})^2 + 4u_x u_{xxx}) + u_{xxxxxx}]_x = 0,$$

and the *seventh-order Sawada–Kotara equation*

$$u_t + [63u^4 + 63(2u^2 u_{xx} + u(u_x)^2) + 21(uu_{xxxx} + (u_{xx})^2 + u_x u_{xxx}) + u_{xxxxxx}]_x = 0;$$

see references in [20, p. 234].

Modern mathematical theory of odd-order quasilinear PDEs is partially originated and continues to be strongly connected with the class of integrable equations. Special advantages of integrability by using the inverse scattering transform method, Lax pairs, Liouville transformations, and other explicit algebraic manipulations made it possible to create rather complete theory for some of these difficult quasilinear PDEs. Nowadays, well-developed theory and most of rigorous results on existence, uniqueness, and various singularity and non-differentiability properties are associated with NDE-type integrable models such as *Fuchssteiner–Fokas–Camassa–Holm* (FFCH) *equation*

$$(I - D_x^2)u_t = -3uu_x + 2u_x u_{xx} + uu_{xxx} \equiv -(I - D_x^2)(uu_x) - \big[u^2 + \tfrac{1}{2}\,(u_x)^2\big]. \tag{1.18}$$

Equation (1.18) is an asymptotic model describing the wave dynamics at the free surface of fluids under gravity. It is derived from Euler equations for inviscid fluids under the long wave asymptotics of shallow water behavior (where the function u is the height of the water above a flat bottom). Applying to (1.18) the integral operator $(I - D_x^2)^{-1}$ with the L^2-kernel $\omega(s) = \frac{1}{2}\,\mathrm{e}^{-|s|} > 0$, reduces it, for a class of solutions, to the conservation law (1.12) with a compact *first-order* perturbation,

$$u_t + uu_x = -\big[\omega * \big(u^2 + \tfrac{1}{2}(u_x)^2\big)\big]_x. \tag{1.19}$$

Almost all mathematical results (including entropy inequalities and Oleinik's condition (E)) have been obtained by using this integral representation of the FFCH equation; see a long list of references given in [20, p. 232].

There is another integrable PDE from the family with third-order quadratic operators,

$$u_t - u_{xxt} = \alpha uu_x + \beta u_x u_{xx} + uu_{xxx} \quad (\alpha,\, \beta \in \mathbb{R}), \tag{1.20}$$

where $\alpha = -3$ and $\beta = 2$ yields the FFCH equation (1.18). This is the *Degasperis–Procesi equation* for $\alpha = -4$ and $\beta = 3$,

$$u_t - u_{xxt} = -4uu_x + 3u_x u_{xx} + uu_{xxx}. \tag{1.21}$$

On existence, uniqueness (of entropy solutions in $L^1 \cap BV$), parabolic ε-regularization, Oleinik's entropy estimate, and generalized PDEs, see [5]. Besides (1.18) and (1.21), the family (1.20) does not contain other integrable entries. A list of more applied papers related to various NDEs is also available in [20, Ch. 4].

1.4. **NDEs from compacton theory.** Other important application of odd-order PDEs are associated with *compacton phenomena* for more general non-integrable models. For instance, the *Rosenau–Hyman* (RH) *equation*

$$u_t = (u^2)_{xxx} + (u^2)_x \tag{1.22}$$

which has special important application as a widely used model of the effects of nonlinear dispersion in the pattern formation in liquid drops [39]. It is the $K(2,2)$ equation from the general $K(m,n)$ family of the following NDEs:

$$u_t = (u^n)_{xxx} + (u^m)_x \quad (u \geq 0), \tag{1.23}$$

that describe phenomena of compact pattern formation, [35, 36]. Such PDEs also appear in curve motion and shortening flows [38]. Similar to well-known parabolic models of porous medium type, the $K(m,n)$ equation (1.23) with $n > 1$ is degenerated at $u = 0$, and therefore may exhibit finite speed of propagation and admit solutions with finite interfaces. The crucial advantage of the RH equation (1.22) is that it possesses *explicit* moving compactly supported soliton-type solutions, called *compactons* [39], which are *travelling wave* (TW) solutions to be discussed for the PDEs under consideration.

Various families of quasilinear third-order KdV-type equations can be found in [4], where further references concerning such PDEs and their exact solutions can be found. Higher-order generalized KdV equations are of increasing interest; see e.g., the quintic KdV equation in [23] and [47], where the seventh-order PDEs are studied. For the $K(2,2)$ equation (1.22), the compacton solutions were constructed in [35].

More general $B(m,k)$ equations, coinciding with the $K(m,k)$ after scaling, $u_t + a(u^m)_x = \mu(u^k)_{xxx}$ also admit simple semi-compacton solutions [40], as well as the $Kq(m,\omega)$ nonlinear dispersion equation (another nonlinear extension of the KdV) [35]

$$u_t + (u^m)_x + [u^{1-\omega}(u^\omega u_x)_x]_x = 0.$$

Setting $m = 2$ and $\omega = \frac{1}{2}$ yields a typical quadratic PDE $u_t + (u^2)_x + uu_{xxx} + 2u_x u_{xx} = 0$ possessing solutions on standard invariant trigonometric-exponential subspaces, where $u(x,t) = C_0(t) + C_1(t)\cos\lambda x + C_2(t)\sin\lambda x$ and $\{C_0, C_1, C_2\}$ solve a nonlinear 3D dynamical system. Combining the $K(m,n)$ and $B(m,k)$ equations gives the dispersive-dissipativity entity $DD(k,m,n)$ [37] $u_t + a(u^m)_x + (u^n)_{xxx} = \mu(u^k)_{xx}$ that can also admit solutions on invariant subspaces for some values of parameters.

For the fifth-order NDEs, such as

$$u_t = \alpha(u^2)_{xxxxx} + \beta(u^2)_{xxx} + \gamma(u^2)_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \tag{1.24}$$

compacton solutions were first constructed in [10], where the more general $K(m,n,p)$ family of PDEs, $u_t + \beta_1(u^m)_x + \beta_2(u^n)_{xxx} + \beta_3 D_x^5(u^p) = 0$, with m, n, $p > 1$, was introduced. Some of these equations will be treated later on. Equation (1.24) is also associated with the family $Q(l,m,n)$ of more general quintic evolution PDEs with nonlinear dispersion,

$$u_t + a(u^{m+1})_x + \omega\big[u(u^n)_{xx}\big]_x + \delta\big[u(u^l)_{xxxx}\big]_x = 0, \tag{1.25}$$

possessing multi-hump, compact solitary solutions [41].

Concerning higher-order in time quasilinear PDEs, let us mention a generalization of the *combined dissipative double-dispersive* (CDDD) *equation* (see, e.g., [33])

$$u_{tt} = \alpha u_{xxxx} + \beta u_{xxtt} + \gamma(u^2)_{xxxxt} + \delta(u^2)_{xxt} + \varepsilon(u^2)_t, \tag{1.26}$$

and also the nonlinear modified dispersive Klein–Gordon equation ($mKG(1,n,k)$),

$$u_{tt} + a(u^n)_{xx} + b(u^k)_{xxxx} = 0, \quad n, k > 1 \quad (u \geq 0); \tag{1.27}$$

see some exact TW solutions in [24]. For $b > 0$, (1.27) is of hyperbolic (or Boussinesq) type in the class of nonnegative solutions. We also mention related 2D *dispersive Boussinesq equations* denoted by $B(m,n,k,p)$ [46],

$$(u^m)_{tt} + \alpha(u^n)_{xx} + \beta(u^k)_{xxxx} + \gamma(u^p)_{yyyy} = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}.$$

See [20, Ch. 4-6] for more references and examples of exact solutions on invariant subspaces of NDEs of various types and orders.

1.5. **On canonical third-order NDEs.** Until recently, quite a little was known about proper mathematics concerning discontinuous solutions, rarefaction waves, and entropy approaches, even for the simplest third-order NDEs such as (1.22) or (see [19])

$$u_t = (uu_x)_{xx}. \tag{1.28}$$

However, the smoothing results for sufficiently regular solutions of linear and nonlinear third-order PDEs are well know from the 1980-90s. For instance, infinite smoothing results were proved in [6] (see also [22]) for the general linear equation

$$u_t + a(x,t)u_{xxx} = 0 \quad (a(x,t) \geq c > 0), \tag{1.29}$$

and in [7] for the corresponding fully nonlinear PDE

$$u_t + f(u_{xxx}, u_{xx}, u_x, u, x, t) = 0 \quad \big(f_{u_{xxx}} \geq c > 0 \big); \tag{1.30}$$

see also [2] for semilinear equations. Namely, for a class of such equation, it is shown that, for data with minimal regularity and sufficient (say, exponential) decay at infinity, there exists a unique solution $u(x,t) \in C_x^\infty$ for small $t > 0$. Similar smoothing local in time results for unique solutions are available for equations in $\mathbb{R}^2$,

$$u_t + f(D^3u, D^2u, Du, u, x, y, t) = 0; \tag{1.31}$$

see [27] and further references therein.

These smoothing results have been used in [16] for developing some δ-entropy concepts for discontinuous solutions by using techniques of smooth deformations. We will follow these ideas applied now to shock and compacton solutions of higher-order NDEs and others.

2. (I) Problem "Blow-up": existence of similarity solutions

We now show that Problem **(I)** on blowing up to the shock $S_-(x)$ can be solved in a unified manner by constructing self-similar solutions. As often happens in nonlinear evolution PDEs, the refined structure of such bounded and generic shocks is described in a scaling-invariant manner.

2.1. **Finite time blow-up formation of the shock wave $S_-(x)$.** One can see that all five NDEs (1.1)–(1.5) admit the following similarity substitution:

$$u_-(x,t) = g(z), \quad z = x/(-t)^{\frac{1}{5}}, \tag{2.1}$$

where, by translation, the blow-up time in reduces to $T=0$. Substituting (2.1) into the NDEs yields for g the following ODEs in $\mathbb{R}$, respectively:

$$gg^{(5)} = -\tfrac{1}{5}\, g'z, \tag{2.2}$$

$$(gg^{(4)})' = -\tfrac{1}{5}\, g'z, \tag{2.3}$$

$$(gg''')'' = -\tfrac{1}{5}\, g'z, \tag{2.4}$$

$$(gg'')''' = -\tfrac{1}{5}\, g'z, \tag{2.5}$$

$$(gg')^{(4)} = -\tfrac{1}{5}\, g'z, \tag{2.6}$$

with the following conditions at infinity for the shocks S_-:

$$g(\mp\infty) = \pm 1. \tag{2.7}$$

In view of the symmetry of the ODEs,

$$\begin{cases} g \mapsto -g, \\ z \mapsto -z, \end{cases} \tag{2.8}$$

it suffices to get odd solutions for $z<0$ posing anti-symmetry conditions at the origin,

$$g(0) = g''(0) = g^{(4)}(0) = 0. \tag{2.9}$$

2.2. **Shock similarity profiles exist and are unique: numerical results.** Before performing a rigorous approach to Problem (I), it is convenient and inspiring to check whether the shock similarity profiles $g(z)$ announced in (2.1) actually exist and are unique for each of the ODEs (2.2)–(2.6). This is done by numerical methods that supply us with positive and convincing conclusions. Moreover, these numerics clarify some crucial properties of profiles that will clarify the actual strategy of rigorous study.

A typical structure of this shock similarity profile $g(z)$ satisfying (2.2), (2.9) is shown in Figure 1. As a key feature, we observe a highly oscillatory behaviour of $g(z)$ about ± 1 as $z \to \mp\infty$, that can essentially affect the topology of the announced convergence (1.7). Therefore, we will need to describe this oscillatory behaviour in detail. In Figure 2, we show the same profile $g(z)$ for smaller z. It is crucial that, in all numerical experiments, we obtained the same profile that indicates that it is the unique solution of (2.2), (2.9).

Figure 3(a)–(d) show the shock similarity profiles for the rest of NDEs (1.2)–(1.5). They differ from each other rather slightly.

Remark: on regularization in numerical methods. For the fifth-order NDEs, this and further numerical constructions are performed by `MatLab` by using the `bvp4c` solver. Typically, we take the relative and absolute tolerances

$$\text{Tols} = 10^{-4}. \tag{2.10}$$

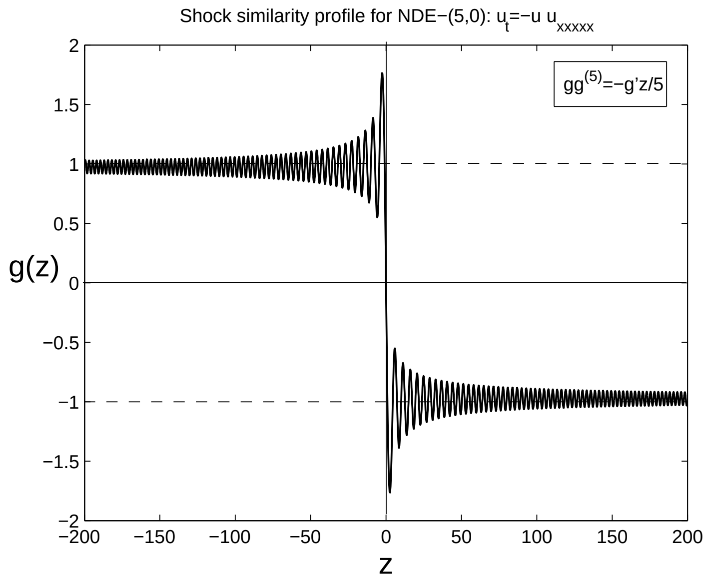


FIGURE 1. The shock similarity profile $g(z)$ as the unique solution of the problem (2.2), (2.9); $z \in [-200, 200]$.

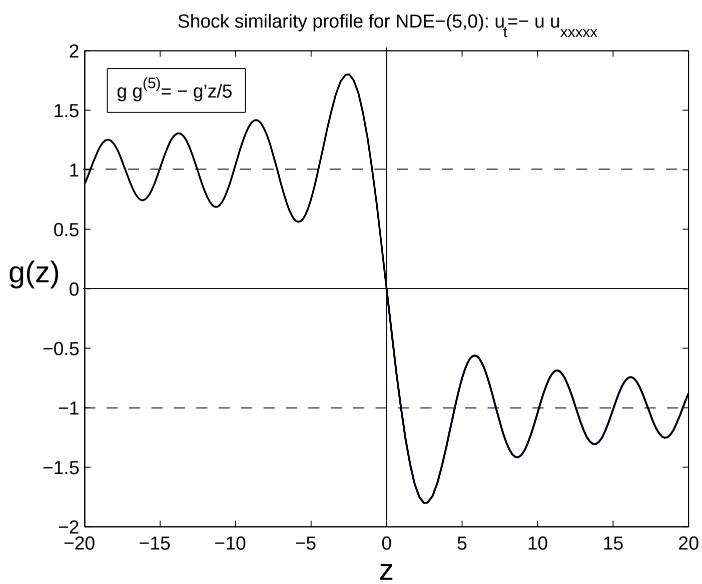
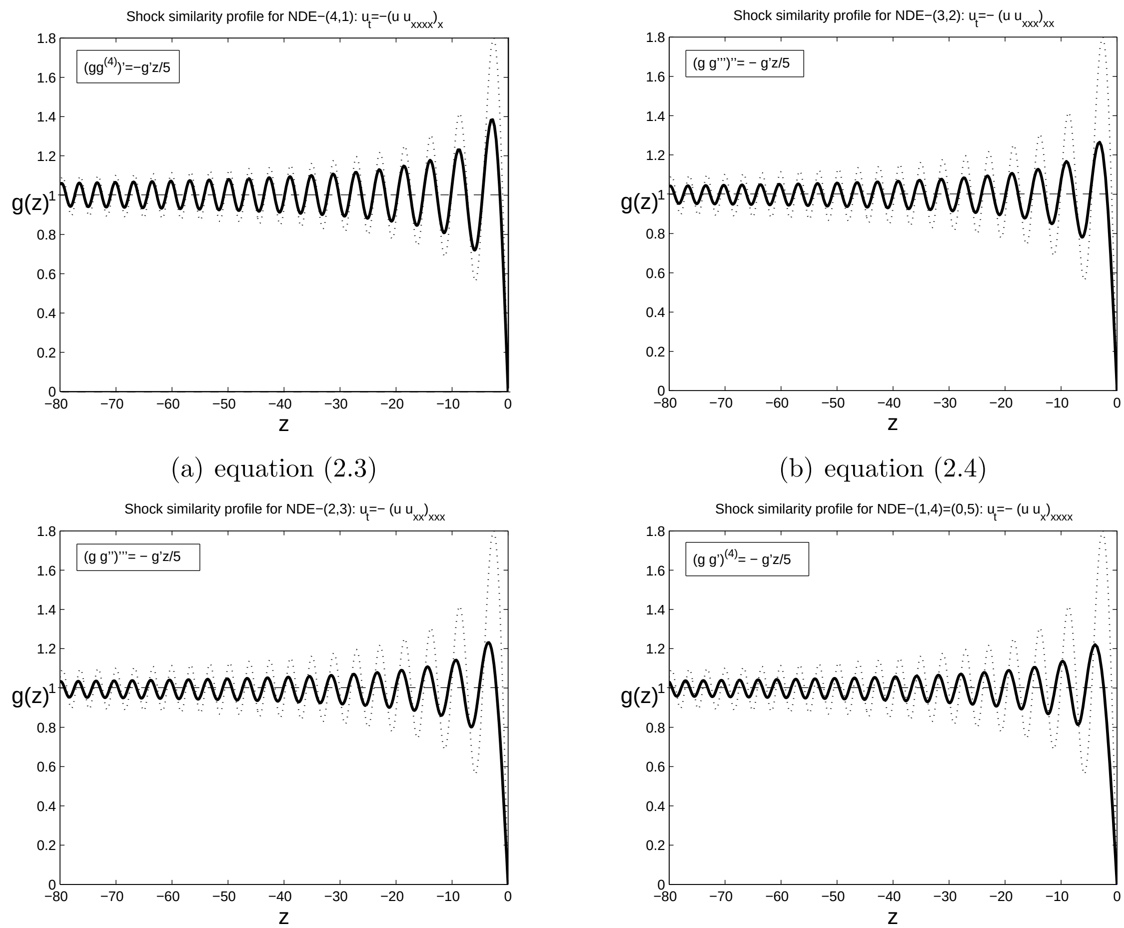


FIGURE 2. The shock similarity profile $g(z)$ as the unique solution of the problem (2.2), (2.9); $z \in [-20, 20]$.

Instead of the degenerate ODE (2.2) (or others), we solve the regularized equation

$$g^{(5)} = -\tfrac{\operatorname{sign} g}{\sqrt{\nu^2+g^2}} \left(\tfrac{1}{5}\, g' z\right), \quad \text{with the regularization parameter } \nu = 10^{-4}, \tag{2.11}$$

where the choice of small ν is coherent with the tolerances in (2.10). Sometimes, we will need to use the enhanced parameters Tols $\sim \nu \sim 10^{-7}$ or even 10^{-9}.



(a) equation (2.3)

(b) equation (2.4)

(c) equation (2.5)

(d) equation (2.6)

FIGURE 3. Shock similarity profiles as solutions of (2.3)–(2.6), (2.9) respectively. For comparison, dotted lines denote the profile from Figures 1 and 2.

2.3. **Justification of oscillatory behaviour about equilibria ± 1 and other asymptotics.** Thus, the shock profiles $g(z)$ are oscillatory about $+1$ as $z \to -\infty$. In order to describe these oscillations in detail, we linearize all the ODEs (2.2)–(2.6) about the regular equilibrium to get the linear ODE

$$g^{(5)} = -\tfrac{1}{5}\, g' z. \tag{2.12}$$

This equation reminds that for the rescaled kernel $F(z)$ of the fundamental solution of the corresponding linear dispersion equation,

$$u_t = -u_{xxxxx} \quad \text{in} \;\; \mathbb{R} \times \mathbb{R}_+. \tag{2.13}$$

The fundamental solution of the corresponding linear operator $\frac{\partial}{\partial t} + D_x^5$ in (2.13) has the standard similarity form

$$b(x,t) = t^{-\frac{1}{5}} F(y), \quad \text{with} \;\; y = x/t^{1/5}, \tag{2.14}$$

where $F(y)$ is a unique solution of the ODE problem

$$\text{(2.15)} \qquad \mathbf{B}_5 F \equiv -F^{(5)} + \tfrac{1}{5}\,(Fy)' = 0 \text{ in } \mathbb{R}, \;\; \int F = 1, \quad \text{or } F^{(4)} = \tfrac{1}{5}\,Fy \text{ on integration.}$$

The precise asymptotics of small solutions of (2.12) as $z \to -\infty$ is as follows: $g(z) \sim \mathrm{e}^{a|z|^{5/4}}$, where $a^4 = \frac{4^4}{5^5}$. Choosing the purely imaginary root with $\mathrm{Re}\, a = 0$ gives a more refined WKBJ-type asymptotics,

$$\text{(2.16)} \qquad g(z) = 1 + C_0 |z|^{-\frac{5}{8}} \big[A \sin\big(a_0 |z|^{\frac{5}{4}}\big) + B \cos\big(a_0 |z|^{\frac{5}{4}}\big) \big] + \dots \, .$$

This asymptotic behaviour implies two conclusions:

$$\text{(2.17)} \qquad g(z) - 1 \notin L^1(\mathbb{R}_-)$$

and that the total variation of $g(z)$ (and hence of $u_-(x,t)$ for any $t<0$) is *infinite*. Setting $|z|^{5/4} = v$ in the integral yields

$$\text{(2.18)} \qquad |g(\cdot)|_{\mathrm{TV}} = \int\limits_{-\infty}^{+\infty} |g'(z)|\,\mathrm{d}z \sim \int\limits^{\infty} z^{-\frac{3}{8}} |\cos z^{\frac{5}{4}}|\,\mathrm{d}z \sim \int\limits^{\infty} \tfrac{|\cos v|}{\sqrt{v}}\,\mathrm{d}v = \infty.$$

This is in striking contrast with the case of conservation laws (1.12), where finite total variation approaches and Helly's second theorem (compact embedding of sets of bounded functions of bounded total variations into L^∞) used to be key; see Oleinik's pioneering approach [30]. In view of the presented properties of the similarity profile $g(z)$, the convergence in (1.7) takes place for any $x \in \mathbb{R}$, uniformly in $\mathbb{R} \setminus (-\mu, \mu)$, $\mu > 0$ small, and in $L^p_{\mathrm{loc}}(\mathbb{R})$ for $p \in [1, \infty)$, that, for convenience, we fix in the following:

Proposition 2.1. *For the shock similarity profile $g(z)$ the convergence (1.7) with $T=0$:*
(i) *does not hold in $L^1(\mathbb{R})$, and*
(ii) *does hold in $L^1_{\mathrm{loc}}(\mathbb{R})$, and moreover, for any fixed finite $l > 0$,*

$$\text{(2.19)} \qquad \|u_-(\cdot,t) - S_-(\cdot)\|_{L^1(-l,l)} = O((-t)^{\frac{1}{8}}) \to 0 \quad \text{as} \quad t \to 0^-.$$

Proof of (2.19) is the same as in (2.18) with a finite interval of integration: for $l = 1$,

$$\|\cdot\|_{L^1(-1,1)} \sim (-t)^{\frac{1}{5}} \int\limits^{(-t)^{-1/5}} z^{-\frac{5}{8}} |\cos z^{\frac{5}{4}}|\,\mathrm{d}z \sim (-t)^{\frac{1}{5}} \int\limits^{(-t)^{-1/4}} v^{-\frac{7}{10}} |\cos v|\,\mathrm{d}z \sim (-t)^{\frac{1}{8}}.$$

Finally, note that each $g(z)$ has a regular asymptotic expansion near the origin. For instance, for the first ODE (2.2), there exist solutions such that

$$\text{(2.20)} \qquad g(z) = Cz + Dz^3 - \tfrac{1}{600}\, z^5 + \tfrac{D}{6300C}\, z^7 + \dots,$$

where $C < 0$ and $D \in \mathbb{R}$ are some constants. The uniqueness of such asymptotics is traced out by using Banach's Contraction Principle applied to the equivalent integral equation in the metric of $C(-\mu, \mu)$, with $\mu > 0$ small. Moreover, it can be shown that (2.20) is the expansion of an analytic function. Other ODEs admit similar local representations of solutions.

In addition, we need the following scaling invariance of the ODEs (2.2)–(2.6): if $g_1(z)$ is a solution, then

$$\text{(2.21)} \qquad g_a(z) = a^5 g_1\big(\tfrac{z}{a}\big) \quad \text{is a solution for any } a \neq 0.$$

2.4. **Existence of shock similarity profiles.** Using the asymptotics derived above, we now in a position to prove the following:

Proposition 2.2. *The problem (2.7), (2.9) for ODEs (2.2)–(2.6) admits a solution $g(z)$, which is an odd analytic function.*

Notice that Figures above clearly convince that, moreover,

$$g(z) > 0 \quad \text{for} \quad z < 0, \tag{2.22}$$

which is difficult to prove rigorously; see further comments below. Actually, (2.22) is not that important for the key convergence (1.7).

Proof. As above, we consider the first ODE (2.2) only. We use the shooting argument using the 2D bundle of asymptotics (2.20). By scaling (2.21), we put $C = -1$, so, actually, we deal with the *one-parameter shooting problem* with the 1D family of orbits satisfying

$$g(z; D) = -z + D\, z^3 - \tfrac{1}{600}\, z^5 - \tfrac{D}{6300}\, z^7 + \ldots, \quad D \in \mathbb{R}. \tag{2.23}$$

It is not hard to check that, besides constant unstable equilibria,

$$g(z) \to C_- > 0 \quad \text{as} \quad z \to -\infty, \tag{2.24}$$

the ODE (2.3) admits an unbounded stable behaviour given by

$$g(z) \sim g_*(z) = -\tfrac{1}{120}\, z^5 \to +\infty \quad \text{as} \quad z \to -\infty. \tag{2.25}$$

This determines the strategy of the 1D shooting via the family (2.23):

(i) obviously, for all $D \ll -1$, we have that $g(z; D) > 0$ is monotone decreasing and approaches the stable behaviour (2.25), and

(ii) on the contrary, for all $D \gg 1$, $g(z; D)$ gets non-monotone and has a zero for some finite $z_0 = z_0(D) < 0$, $z_0(D) \to 0^-$ as $D \to -\infty$, and eventually approaches (2.25), but in an essentially *non-monotone* way.

It follows from different and opposite "topologies" of the behaviour announced in (i) and (ii) that there exists a constant D_0 such that $g(z; D_0)$ does not belong to those two sets of orbits (both are open) and hence does not approach $g_*(z)$ as $z \to -\infty$ at all. This is precisely the necessary shock similarity profile with some unknown properties. $\square$

This 1D shooting approach is explained in Figure 4 obtained numerically, where

$$D_0 = 0.069192424\ldots\,. \tag{2.26}$$

It seems that as $D \to D_0^+$, the zero of $g(z; D)$ must disappear at infinity, i.e.,

$$z_0(D) \to -\infty \quad \text{as} \quad D \to D_0^+, \tag{2.27}$$

and this actually happens as Figure 4 shows. Then this would justify the positivity (2.22). Unfortunately, in general (i.e., for similar ODEs with different sufficiently arbitrary nonlinearities), this is not true, i.e., cannot be guaranteed by a topological argument. So that the actual operator structure of the ODEs should be involved, so, theoretically, the positivity is difficult to guarantee in general. Note again that, if the shock similarity profile $g(z)$ would have a few zeros for $z < 0$, this would not affect the crucial convergence property such as (1.7).

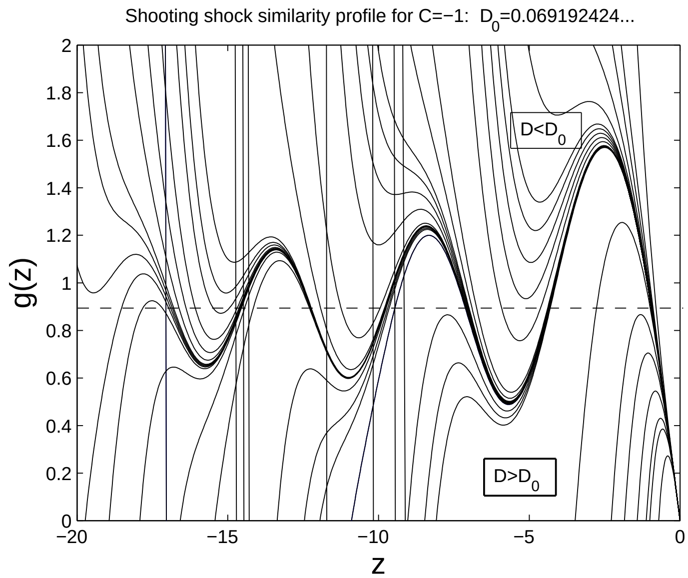


FIGURE 4. Shooting the shock similarity profile $g(z)$ via the family (2.23); $D_0 = 0.069192424...$.

2.5. **Self-similar formation of other shocks.** *NDE*–$(1,4)$. Let us first briefly consider the last ODE (2.6) for the fully divergent NDE (1.5). Similarly, by the same arguments, we show that, according to (2.1), there exist other shocks as non-symmetric step-like functions, so that, as $t \to 0^-$,

$$u_-(x,t) \to \begin{cases} C_- > 0 & \text{for } x < 0, \\ C_0 & \text{for } x = 0, \\ C_+ < 0 & \text{for } x > 0, \end{cases} \tag{2.28}$$

where $C_- \ne -C_+$ and $C_0 \ne 0$. Figure 5 shows a few of such similarity profiles $g(z)$, where three of these are strictly positive. The most interesting is the boldface one with $C_- = 1.4$ and $C_+ = 0$ that has the finite right-hand interface at $z = z_0 \approx 5$, with the expansion

$$g(z) = -\tfrac{z_0}{4200}\,(z_0 - z)_+^4(1 + o(1)) \to 0^- \text{ as } z \to z_0. \tag{2.29}$$

It follows that this $g(z) < 0$ near the interface so the function changes sign there, which is also seen in Figure 5 by carefully checking the shape of profiles above the boldface one with the finite interface bearing in mind a natural continuous dependence on parameters.

NDE–$(5,0)$. Consider next the first ODE (2.2) for the fully non-divergent NDE–$(5,0)$ (1.1). We again can describe formation of shocks (2.28); see Figure 6. The boldface profile with $C_- = 1.4$ and $C_+ = 0$ has finite right-hand interface at $z = z_0 \approx 5$, with a different expansion

$$g(z) = \tfrac{6z_0}{5}\,(z_0 - z)^4 |\ln(z_0 - z)|(1 + o(1)) \to 0^+ \text{ as } z \to z_0^-. \tag{2.30}$$

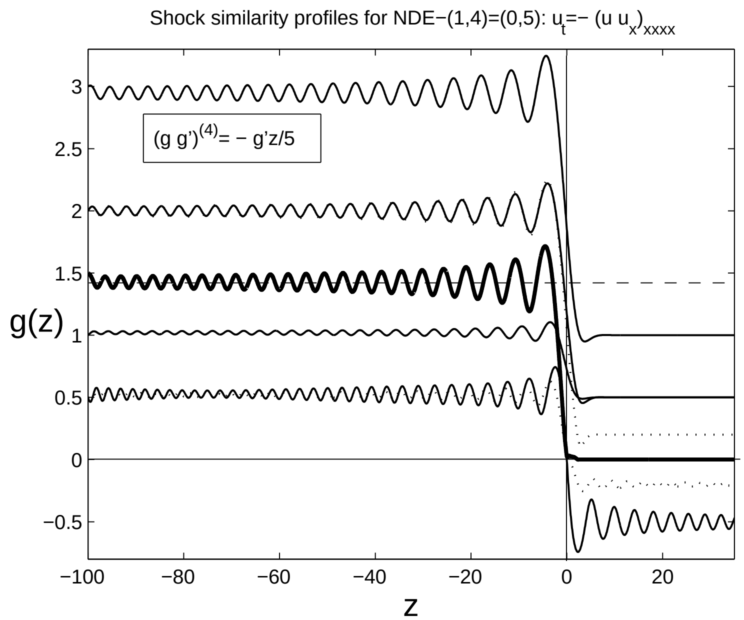


FIGURE 5. Various shock similarity profiles $g(z)$ as solutions of the problem (2.6), (2.9).

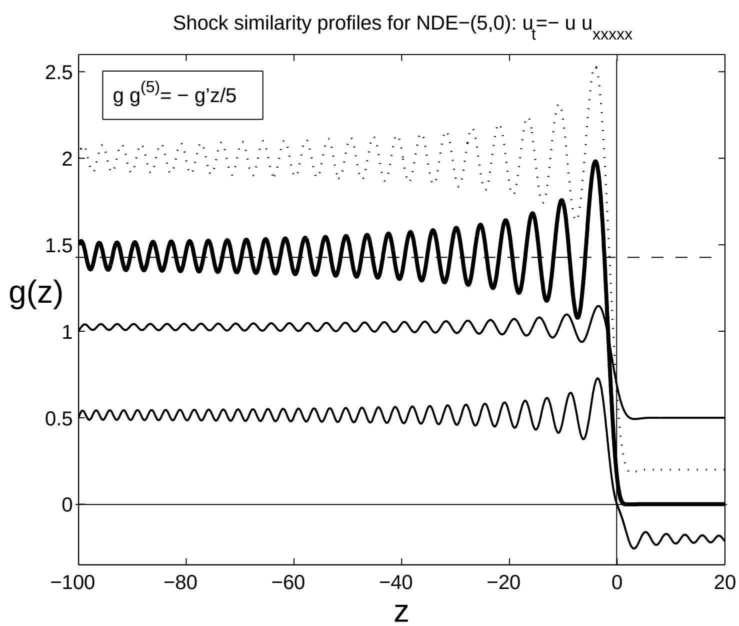


FIGURE 6. Various shock similarity profiles $g(z)$ as solutions of the problem (2.2), (2.9).

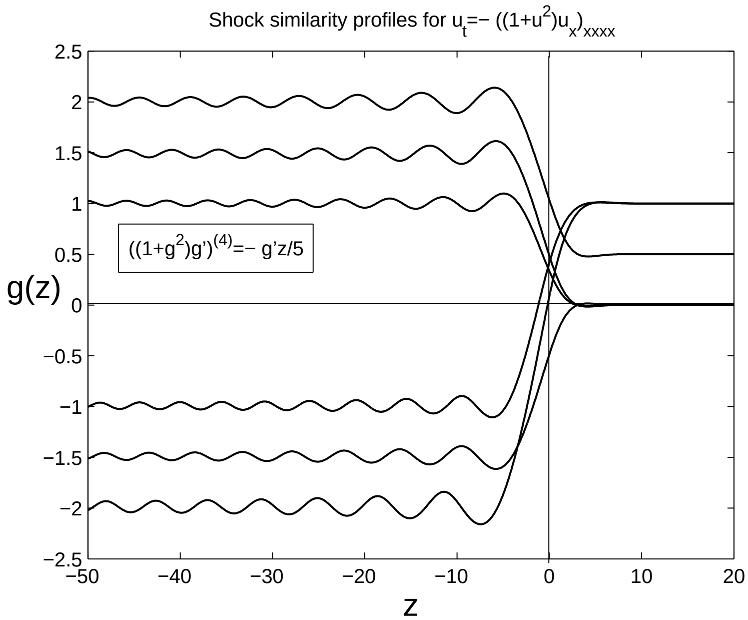


FIGURE 7. Various shock similarity profiles $g(z)$ satisfying the ODE (2.32).

2.6. **On shock formation for a uniform NDE.** Here, as a key example (to be continued), we show shocks for uniform NDEs such as the fully divergent one

$$u_t = -((1+u^2)u_x)_{xxxx}. \tag{2.31}$$

This equation is non-degenerate, so represents a "uniformly dispersive" equation. The ODE for self-similar solutions (2.1) then takes the form

$$((1+g^2)g')^{(4)} = -\tfrac{1}{5}\, g'z\,. \tag{2.32}$$

The mathematics of such equations is similar to that in Section 2.4. In Figure 7, we present a few shock similarity profiles for (2.32). Note that both shocks $S_\pm(x)$ are admissible, since for the ODE (and the NDE (2.31)), we have, instead of symmetry (2.8),

$$-\, g(z) \quad \text{is also a solution.} \tag{2.33}$$

3. **Riemann Problem** S_+: SIMILARITY RAREFACTION WAVES

Using the reflection symmetry of all the NDEs under consideration,

$$\begin{cases} u \mapsto -u, \\ t \mapsto -t, \end{cases} \tag{3.1}$$

we conclude that these admit *global* similarity solutions defined for all $t > 0$,

$$u_+(x,t) = g(z), \;\; \text{with } z = x/t^{\frac{1}{5}}. \tag{3.2}$$

Then $g(z)$ solves the ODEs (2.2)–(2.6) with the opposite terms

$$... = \tfrac{1}{5}\, g'z \tag{3.3}$$

on the right-hand side. The conditions (2.7) also take the opposite form

$$f(\pm\infty) = \pm 1. \tag{3.4}$$

Thus, these profiles are obtained from the blow-up ones in (2.1) by reflection,

(3.5) if $g(z)$ is a blow-up shock profile in (2.1), then $g(-z)$ is rarefaction one in (3.2).

These are sufficiently regular similarity solutions of NDEs that have the necessary initial data: by Proposition 2.1(ii), in $L^1_{\rm loc}$,

$$u_+(x,t) \to S_+(x) \quad \text{as} \quad t \to 0^+. \tag{3.6}$$

Other profiles $g(-z)$ from shock wave similarity patterns generate further rarefaction solutions including those with finite left-hand interfaces.

4. **Riemann Problem** S_-: towards δ-entropy test

4.1. **Uniform NDEs.** In this section, for definiteness, we consider the fully non-divergent NDE (1.1),

$$u_t = \mathbf{A}(u) \equiv -u u_{xxxxx} \quad \text{in} \quad \mathbb{R} \times (0,T), \quad u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R}). \tag{4.1}$$

In order to concentrate on shocks and to avoid difficulties with finite interfaces or transversal zeros at which $u = 0$ (these are weak discontinuities via non-uniformity of the PDE), we deal with strictly positive solutions, say,

$$\tfrac{1}{C} \le u \le C, \quad \text{where} \quad C > 1 \quad \text{is a constant.} \tag{4.2}$$

Remark: uniformly non-degenerate NDEs. Alternatively, in order to avoid the assumptions like (4.2), we can consider the uniform equations such as, e.g.,

$$u_t = -(1+u^2)u_{xxxxx}, \tag{4.3}$$

for which no finite interfaces are available. Of course, (4.3) admits similar blow-up similarity formation of shocks by (2.1). In Figure 8, we show a few profiles satisfying

$$(1+g^2)g^{(5)} = -\tfrac{1}{5}\, g' z, \quad z \in \mathbb{R}. \tag{4.4}$$

Recall that, for (4.4), (2.33) holds, so both $S_\pm(x)$ are admissible and entropy (see below).

4.2. **On uniqueness, continuous dependence, and *a priori* bounds for smooth solutions.** Actually, in our δ-entropy construction, we will need just a local semigroup of smooth solutions that is continuous is $L^1_{\rm loc}$. The fact that such results are true for fifth-order (or other odd-order NDEs) is easy to illustrated as follows.

One can see that, since (4.1) is a dispersive equation, which contains no dissipative terms, the uniqueness follows as for parabolic equations such as

$$u_t = -u u_{xxxx} \quad \text{or} \quad u_t = u u_{xxxxxx} \quad \big(\text{in the class } \big\{\tfrac{1}{C} \le u \le C\big\}\big).$$

Thus, we assume that $u(x,t)$ solves (4.1) with initial data $u_0(x) \in H^{10}(\mathbb{R})$, satisfies (4.2), and is sufficiently smooth, $u \in L^\infty([0,T], H^{10}(\mathbb{R}))$, $u_t \in L^\infty([0,T], H^5(\mathbb{R}))$, etc.

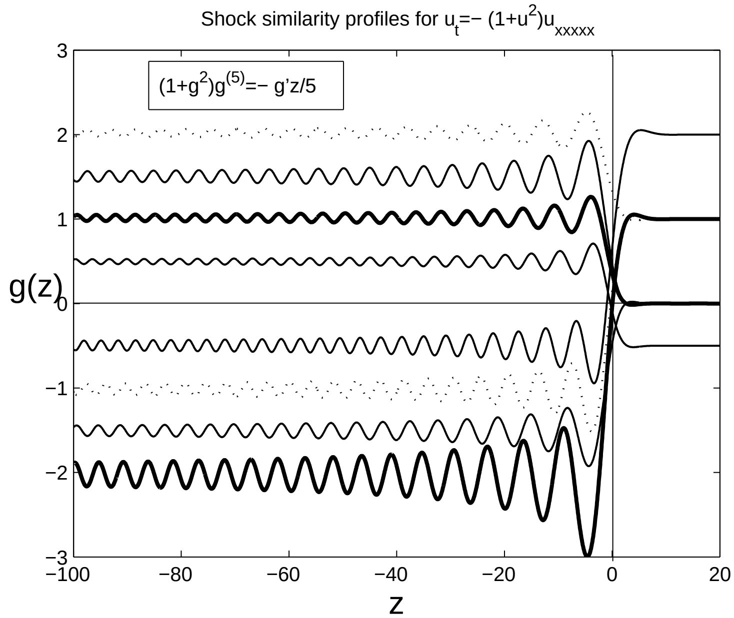


FIGURE 8. Various shock similarity profiles $g(z)$ satisfying the ODE (4.4).

Assuming that $v(x,t)$ is the second smooth solution, we subtract equations to obtain for the difference $w = u - v$ the PDE

$$w_t = -uw_{xxxxx} - v_{xxxxx}w. \tag{4.5}$$

We next divide by $u \geq \frac{1}{C} > 0$ and multiply by w in L^2, so integrating by parts that vanishes the dispersive term w_{xxxxx} yields

$$\int \frac{ww_t}{u} \equiv \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{w^2}{u} + \frac{1}{2} \int \frac{u_t}{u^2}\, w^2 = -\int \frac{v_{xxxxx}}{u}\, w^2. \tag{4.6}$$

Therefore, using (4.2) and the assumed regularity yields

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{w^2}{u} = \int \Big(-\frac{1}{2}\,\frac{u_t}{u^2} - \frac{v_{xxxxx}}{u} \Big)\, w^2 \leq C_1 \int \frac{w^2}{u}, \tag{4.7}$$

where the derivatives $u_t(\cdot,t)$ and $v_{xxxxx}(\cdot,t)$ are from $L^\infty([0,T])$. By Gronwall's inequality, (4.7) yields $w(t) \equiv 0$. Obviously, these estimates can be translated to the continuous dependence result in L^2 and hence in L^1_{loc}.

Other *a priori* bounds on solutions can be also derived in the lines of computations in [7, §§ 2, 3] that lead to rather technical manipulations. The principal fact is the same as seen from (4.7): differentiating α times in x equation (4.1) and setting $v = D_x^\alpha u$ yields the equations with the same as in (4.5) principal part:

$$v_t = -uv_{xxxxx} + \ldots\,. \tag{4.8}$$

Multiplying this by $\zeta \frac{v}{u}$, with ζ being a cut-off function, and using various interpolation inequalities makes it possible to derive necessary *a priori* bounds and hence to observe the corresponding smoothing phenomenon for exponentially decaying initial data.

4.3. **On local semigroup of smooth solutions of uniform NDEs and linear operator theory.** We recall that local C^∞-smoothing phenomena are known for third-order linear and fully nonlinear dispersive PDEs; see [6, 7, 22, 27] and earlier references therein. We claim that, having obtained *a priori* bounds, a smooth local solution can be constructed by the iteration techniques as in [7, § 3] by using a standard scheme of iteration of the equivalent integral equation for spatial derivatives. We present further comments concerning other approaches to local existence, where we return to integral equations.

We then need a detailed spectral theory of fifth-order operators such as

$$\mathbf{P}_5 = a(x)D_x^5 + b(x)D_x^4 + ... , \quad x \in (-L, L) \quad \big(a(x) \ge \tfrac{1}{C} > 0\big), \tag{4.9}$$

with bounded coefficients. This theory is developed in Naimark's book [29, Ch. 2]. For *regular boundary conditions* (e.g., for periodic ones that are regular for any order and that suit us well), operators (4.9) admit a discrete spectrum $\{\lambda_k\}$, where the eigenvalues λ_k are all simple for all large k.

It is crucial for further use of eigenfunction expansion techniques that the complete in L^2 subset of eigenfunctions $\{\psi_k\}$ creates a *Riesz basis*, i.e., for any $f \in L^2$,

$$\textstyle\sum |\langle f, \psi_k\rangle|^2 < \infty, \quad \text{where} \quad \langle f, \psi_k\rangle = \int f\, \overline{\psi}_k, \tag{4.10}$$

and, for any $\{c_k\} \in l^2$ $\big($i.e., $\sum |c_k|^2 < \infty\big)$, there exists a function $f \in L^2$ such that

$$\langle f, \psi_k\rangle = c_k. \tag{4.11}$$

Then there exists the unique set of "adjoint" generalized eigenfunctions $\{\psi_k^*\}$ (attributed to the"adjoint" operator $\mathbf{P}_5^*$) being also a Riesz basis that is bi-orthonormal to $\{\psi_k\}$:

$$\langle \psi_k, \psi_l^*\rangle = \delta_{kl}. \tag{4.12}$$

Hence, for any $f \in L^2$, in the sense of the mean convergence,

$$f = \textstyle\sum c_k \psi_k, \quad \text{with} \quad c_k = \langle f, \psi_k^*\rangle. \tag{4.13}$$

See further details in [29, § 5].

The eigenvalues of (4.9) have the asymptotics

$$\lambda_k \sim (\pm 2\pi k \mathrm{i})^5 \quad \text{for all} \quad k \gg 1. \tag{4.14}$$

In particular, it is known that $\mathbf{P}_5$ has compact resolvent that makes it possible to use it in the integral representation of the NDEs; cf. [7, § 3], where integral equations are used to construct a unique smooth solution of third-order NDEs.

On the other hand, this means that $\mathbf{P}_5 - aI$ for any $a \gg 1$ is not a sectorial operator that makes suspicious using advanced theory of analytic semigroups [8, 13, 28], which is natural for even-order parabolic flows; see further discussion below. Analytic smoothing effect for higher-order dispersive equations were studied in [43]. Concerning unique continuation and continuous dependence properties for dispersive equations, see [11] and references therein, and also [44] for various estimates.

For the linear dispersion equation with constant coefficients (2.13), the Cauchy problem with integrable data $u_0(x)$ admits the unique solution

$$u(x,t) = b(x-\cdot,t) * u_0(\cdot), \tag{4.15}$$

where $b(x,t)$ is the fundamental solution (2.14). Analyticity of solutions in t (and x) can be associated with the rescaled operator

$$\mathbf{B}_5 = -D_z^5 + \tfrac{1}{5}\, zD_z + \tfrac{1}{5}\, I \quad \text{in} \quad L^2_\rho(\mathbb{R}), \quad \rho(z) = \mathrm{e}^{a|z|^{5/4}}, \tag{4.16}$$

where $a > 0$ is a sufficiently small constant. Here, $\mathbf{B}_5$ in (4.16) is the operator in (2.15) that generates the rescaled kernel F of the fundamental solution in (2.14). Namely, using the same rescaling as in (2.14), we set

$$u(x,t) = t^{-\frac{1}{5}} v(y,\tau), \quad y = x/t^{\frac{1}{5}}, \quad \tau = \ln t, \tag{4.17}$$

to get the rescaled PDE with the operator (4.16),

$$v_\tau = \mathbf{B}_5 v, \tag{4.18}$$

so that (4.15), on Taylor's expansion of the kernel, yields

$$v(y,\tau) = \textstyle\int F(y - z\mathrm{e}^{-\tau/5})\, u_0(z)\, \mathrm{d}z = \sum\limits_{(k)} \frac{(-1)^k}{\sqrt{k!}}\, F^{(k)}(y) \mathrm{e}^{-\tau/5} \frac{1}{\sqrt{k!}} \textstyle\int z^k u_0(z)\, \mathrm{d}z, \tag{4.19}$$

where the series converges uniformly on compact subsets thus defining an analytic solution. Then (4.16) has the real spectrum and the eigenfunctions (see [15, § 9])

$$\sigma(\mathbf{B}_5) = \big\{ -\tfrac{l}{5},\, l = 0,1,2,... \big\} \quad \text{and} \quad \psi_k(y) = \tfrac{(-1)^k}{\sqrt{k!}}\, F^{(k)}(y), \;\; k \ge 0.$$

See a "parabolic" version of such a spectral theory in [12]. This suggests that $\mathbf{B}_5 - aI$ is sectorial for $a \ge 0$ ($\lambda_0 = 0$ is simple), and this justifies the fact that (4.15) is an analytic (in t) flow. Let us mention again that analytic smoothing effects are known for higher-order dispersive equations with operators of principal type, [43].

Actually, this suggests to treat (4.1), (4.2) by classic approach as in Da Prato–Grisvard [8] by linearizing about a sufficiently smooth $u_0 = u(t_0)$, $t_0 \ge 0$, by setting $u(t) = u_0 + v(t)$ giving the linearized equation

$$v_t = \mathbf{A}'(u_0)v + \mathbf{A}(u_0) + g(v), \quad t > t_0; \quad v(t_0) = 0, \tag{4.20}$$

where $g(v)$ is a quadratic perturbation. Using the good semigroup $\mathrm{e}^{\mathbf{A}'(u_0)t}$, this makes it possible to study local regularity properties of the corresponding integral equation

$$v(t) = \int\limits_{t_0}^{t} \mathrm{e}^{\mathbf{A}'(u_0)(t-s)} (\mathbf{A}(u_0) + g(v(s)))\, \mathrm{d}s. \tag{4.21}$$

Note that this smoothing approach demands a fast exponential decay of solutions $v(x,t)$ as $x \to \infty$, since one needs that $v(\cdot,t) \in L^2_\rho$; cf. [27], where C^∞-smoothing for third-order NDEs was also established under the exponential decay. Equation (4.21) can be used to guarantee local existence of smooth solutions of a wide class of odd-order NDEs.

Thus, we state the following conclusion to be used later on:

$$\text{any sufficiently smooth solution } u(x,t) \text{ of (4.1), (4.2) at } t=t_0, \\ \text{can be uniquely extended to some interval } t \in (t_0, t_0+\nu),\ \nu>0. \tag{4.22}$$

4.4. **Smooth deformations and δ-entropy test for solutions with shocks.** The situation dramatically changes if we want to treat solutions with shocks. Namely, it is known that even for the NDE–3 (1.28), the similarity formation mechanism of shocks shows nonuniqueness extension of solutions after a typical "gradient" catastrophe, [17]. Therefore, we dot a chance to get in an easy (or any) manner a uniqueness/entropy result for more complicated NDEs such as (1.5) by using the δ-deformation (evolutionary smoothing) approach. However, we will continue using these fruitful ideas in order to develop a much weaker *"δ-entropy test"* for distinguishing various shock and rarefaction waves.

Thus, given a small $\delta>0$ and a sufficiently small bounded continuous (and, possibly, compactly supported) solution $u(x,t)$ of the Cauchy problem (4.1), satisfying (4.2), we construct its smooth *δ-deformation*, aiming get smoothing in a small neighbourhood of bounded shocks, as follows. Note that we deal here with simple shock configurations (mainly, with 1-shock structures), and do not aim to cover more general shock geometry, which can be very complicated; especially since we do not know all types of simple single-point moving shocks.

(i) We perform a smooth δ-deformation of initial data $u_0(x)$ by introducing a suitable C^1 function $u_{0\delta}(x)$ such that

$$\int |u_0 - u_{0\delta}| < \delta. \tag{4.23}$$

If u_0 is already sufficiently smooth, this step is abandoned (now and later on). By $u_{1\delta}(x,t)$, we denote the unique local smooth solution of the Cauchy problem with data $u_{0\delta}$, so that, by (4.22), continuous function $u_{1\delta}(x,t)$ is defined on the maximal interval $t\in[t_0,t_1(\delta))$, where we denote $t_0=0$ and $t_1(\delta)=\Delta_{1\delta}$. At this step, we are able to eliminate non-evolution (evolutionary unstable) shocks, which then create corresponding smooth rarefaction waves.

(ii) Since at $t=\Delta_{1\delta}$, a shock-type discontinuity (or possibly infinitely many shocks) is supposed to occur, since otherwise we extend the continuous solution by (4.22), we perform another suitable δ-deformation of the "data" $u_{1\delta}(x,\Delta_{1\delta})$ to get a unique continuous solution $u_{2\delta}(x,t)$ on the maximal interval $t\in[t_1(\delta),t_2(\delta))$, with $t_2(\delta)=\Delta_{1\delta}+\Delta_{2\delta}$, etc. Here and in what follows, we always mean a "δ-smoothing" performed in a small neighbourhood of occurring singularities ONLY (shock waves).

...

(k) With suitable choices of each δ-deformations of "data" at the moments $t=t_j(\delta)$, when $u_{j\delta}(x,t)$ has a shock, there exists a $t_k(\delta)>1$ for some finite $k=k(\delta)$, where $k(\delta)\to+\infty$ as $\delta\to 0$. It is easy to see that, for bounded solutions, $k(\delta)$ is always finite. A contradictions is obtained while assuming that $t_j(\delta)\to\bar t<1$ as $j\to\infty$ for arbitrarily

small $\delta > 0$ meaning a kind of "complete blow-up" that was excluded by assumption of smallness of the data.

This gives a *global δ-deformation* in $\mathbb{R} \times [0,1]$ of the solution $u(x,t)$, which is the discontinuous orbit denoted by

$$u^\delta(x,t) = \{u_{j\delta}(x,t) \ \text{ for } \ t \in [t_{j-1}(\delta), t_j(\delta)), \quad j = 1, 2, ..., k(\delta)\}. \tag{4.24}$$

One can see, this δ-deformation construction aims checking a kind of *evolution stability* of possible shock wave singularities and therefore, to exclude those that are not entropy and evolutionary generate smooth rarefaction waves.

Finally, by an arbitrary *smooth δ-deformation*, we will mean the function (4.24) constructed by any sufficiently refined finite partition $\{t_j(\delta)\}$ of $[0,1]$, without reaching a shock of S_--type at some or all intermediate points $t = t_j^-(\delta)$.

We next say that, given a solution $u(x,t)$, it is *stable relative smooth deformations*, or simply δ-stable (*δeformation-stable*), if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any finite δ-deformation of u given by (4.24),

$$\iint |u - u^\delta| < \varepsilon. \tag{4.25}$$

Recall that (4.24) is an δ-orbit, and, in general, is not and cannot be aimed to represent a fixed solution in the limit $\delta \to 0$; see below.

4.5. **On δ-entropy solutions.** Having checked that the local smooth solvability problem above is well-posed, we now present the corresponding definition that will be applied to particular weak solutions. Recall that the metric of convergence, $L^1_{\rm loc}$ in present, for (1.28) was justified by a similarity analysis presented in Proposition 2.1. For other types of shocks and/or NDEs, the topology may be different.

Thus, under the given hypotheses, a function $u(x,t)$ is called a δ-entropy solution of the Cauchy problem (4.1), if there exists a sequence of its smooth δ-deformations $\{u^{\delta_k},\, k = 1, 2, ...\}$, where $\delta_k \to 0$, which converges in $L^1_{\rm loc}$ to u as $k \to \infty$.

This is slightly weaker (but equivalent) to the condition of δ-stability.

4.6. **δ-entropy test and on nonexistent uniqueness.** Since, by obvious reasons, the δ-deformation construction gets rid of non-evolution shocks (leading to non-singular rarefaction waves), a first consequence of the construction is that it defines the *δ-entropy test* for solutions, which allows one, at least, to distinguish the true shocks from artificial rarefaction wave.

In Section 5, we show that it is completely non-realistic to expect something essentially more from this construction in the direction of uniqueness and/or entropy-like selection of proper solutions. Though these expectations well-correspond to previous classic PDE entropy-like theories, these are too much excessive for higher-order models, where such a universal property is not achievable at all anymore. Even proving convergence for a fixed special δ-deformation is not easy at all. Thus, for particular cases, we will use the above notions with convergence along a subsequence of δ's to classify and distinguish shocks and rarefaction waves:

4.7. **First easy conclusions of δ-entropy test.** As a first application, we have:

Proposition 4.1. *Shocks of the type $S_-(x)$ are δ-entropy for (4.1).*

The result follows from the properties of similarity solutions (2.1), which, by varying the blow-up time $T \mapsto T+\delta$, can be used as their local smooth δ-deformations at any point $t \in [0,1)$.

Proposition 4.2. *Shocks of the type $S_+(x)$ are not δ-entropy for (4.1).*

Indeed, taking initial data $S_+(x)$ and constructing its smooth δ-deformation via the self-similar solution (3.2) with shifting $t \mapsto t+\delta$, we obtain the global δ-deformation $\{u^\delta = u_+(x, t+\delta)\}$, which goes away from S_+.

Thus, the idea of smooth δ-deformations allows us to distinguish basic δ-entropy and non-entropy shocks *without* any use of mathematical manipulations associated with standard entropy inequalities, which, indeed, are illusive for higher-order NDEs; cf. [17].

5. On nonuniqueness after shock formation

Here we mainly follow the ideas from [17] applied to the NDE–3 (1.28), so we will omit some technical data and present more convincing analytic and numerical results (numerics are essential for occurred hard 5D dynamical systems). Without loss of generality, we always deal with the divergence NDE–5 (1.5).

5.1. **Main strategy towards nonunique continuation.** We begin with the study of new shock patterns, which are induced by other similarity solutions of (1.5):

$$u_-(x,t) = (-t)^\alpha f(y), \quad y = \tfrac{x}{(-t)^\beta}, \quad \beta = \tfrac{1+\alpha}{5}, \quad \text{where } \alpha \in \left(0, \tfrac{1}{4}\right) \text{ and} \tag{5.1}$$

$$\begin{cases} -(ff')^{(4)} - \beta f'y + \alpha f = 0 \quad \text{in} \quad \mathbb{R}_-, \quad f(0) = f''(0) = f^{(4)}(0) = 0, \\ f(y) = C_0|y|^{\frac{\alpha}{\beta}}(1+o(1)) \quad \text{as} \quad y \to -\infty, \quad C_0 > 0. \end{cases} \tag{5.2}$$

In this section, in order to match the key results in [17], in (2.1) and later on, we change the variables $\{g, z\} \mapsto \{f, y\}$. In the next Section 6, we return to the original notation.

The anti-symmetry conditions in (5.2) allow to extend the solution to the positive semi-axis $\{y > 0\}$ by $-f(-y)$ to get a global pattern.

Obviously, the solutions (2.1), which are suitable for Riemann's problems, correspond to the simple case $\alpha = 0$ in (5.1). It is easy to see that, for positive α, the asymptotics in (5.2) ensures getting first *gradient blow-up* at $x = 0$ as $t \to 0^-$, as a weak discontinuity, where the final time profile remains locally bounded and continuous:

$$u_-(x, 0^-) = \begin{cases} C_0|x|^{\frac{\alpha}{\beta}} & \text{for} \quad x < 0, \\ -C_0|x|^{\frac{\alpha}{\beta}} & \text{for} \quad x > 0, \end{cases} \tag{5.3}$$

where $C_0 > 0$ is an arbitrary constant. Note that the standard "gradient catastrophe", $u_x(0, 0^-) = -\infty$, then occurs in the range, which we will deal within,

$$\tfrac{\alpha}{\beta} < 1 \quad \text{provided that} \quad \alpha < \tfrac{1}{4}. \tag{5.4}$$

Thus, the wave braking (or "overturning") begins at $t = 0$, and next we show that it is performed again in a self-similar manner and is described by similarity solutions

$$u_+(x,t) = t^\alpha F(y), \quad y = \tfrac{x}{t^\beta}, \quad \beta = \tfrac{1+\alpha}{5}, \quad \text{where} \tag{5.5}$$

$$\begin{cases} -(FF')^{(4)} + \beta F' y - \alpha F = 0 \quad \text{in} \quad \mathbb{R}_-, \\ F(0) = F_0 > 0, \;\; F(y) = C_0 |y|^{\frac{\alpha}{\beta}}(1+o(1)) \quad \text{as} \quad y \to -\infty, \end{cases} \tag{5.6}$$

where the constant $C_0 > 0$ is fixed by blow-up data (5.3). The asymptotic behaviour as $y \to -\infty$ in (5.6) guarantees the continuity of the global discontinuous pattern (with $F(-y) \equiv -F(y)$) at the singularity blow-up instant $t = 0$, so that

$$u_-(x, 0^-) = u_+(x, 0^+) \quad \text{in} \quad \mathbb{R}. \tag{5.7}$$

Then any suitable couple $\{f, F\}$ defines a global solution $u_\pm(x,t)$, which is continuous at $t = 0$, and then it is called an *extension pair*. It was shown in [17] that, for typical NDEs–3, the pair is not uniquely determined and there exist infinitely many shock-type extensions of the solution after blow-up at $t = 0$. We are going to describe a similar non-uniqueness phenomenon for the NDEs–5 such as (1.5).

A first immediate consequence of our similarity blow-up/extension analysis is as follows:

(5.8) in the CP, formation of shocks for the NDE (1.5) can lead to nonuniqueness.

The second conclusion is more subtle and is based on the fact that, for some initial data at $t = 0$, the solution set for $t > 0$ does not contain any "minimal", "maximal", or "extremal" points in any reasonable sense, which might play a role of a unique "entropy" one chosen by introducing a hypothetical entropy inequalities, conditions, or otherwise. If this is true for the whole set of such weak solutions of (1.5) with initial data (5.3), then, for the Cauchy problem,

(5.9) there exists no general "entropy mechanism" to choose a unique solution.

Actually, overall, (5.8) and (5.9) show that the problem of uniqueness of weak solutions for the NDEs such as (1.5) cannot be solved in principal. On the other hand, in a FBP setting by adding an extra suitable condition on shock lines, the problem might be well-posed with a unique solution, though proofs can be very difficult. We refer again to more detailed discussion of these issues for the NDE–3 (1.28) in [17]. Though we must admit, that for the NDE–5 (1.5), which induces 5D dynamical systems for the similarity profiles (and hence 5D phase spaces), those nonuniqueness and non-entropy conclusions are not that clear and some of their aspects do unavoidably remain questionable and open.

Hence, the *non-uniqueness* in the CP is a non-removable issue of PDE theory for higher-order degenerate nonlinear odd-order equations (and possibly not only those). The non-uniqueness of solutions of (1.5) has a pure dimensional nature associated with the structure of the 5D phase space of the ODE (5.6).

5.2. **Gradient blow-up similarity solutions.** Consider the blow-up ODE problem (5.2), which is a difficult one, with a *5D phase space*. Note that, by invariant scaling (2.21), it can be reduced to a 4th-order ODE with a too complicated nonlinear operator composed from too many polynomial terms, so we do not rely on that and work in the original phase space. Therefore, some more delicate issues on, say, uniqueness of certain orbits, become very difficult or even remain open, though some more robust properties can be detected rigorously. We will also use numerical methods for illustrating and even justifying some of our conclusions. For the fifth-order equations such as (5.2), this and further numerical constructions are performed by the `MatLab` with the standard `ode45` solver therein.

Let us describe the necessary properties of orbits $\{f(y)\}$ we are interested in. Firstly, it follows from the conditions in (5.2) that, for $y \approx 0^-$,

(5.10) the set of proper orbits is 2D parameterized by $f_1 = f'(0) < 0$ and $f_3 = f'''(0)$.

Secondly and on the other hand, the necessary behaviour at infinity is as follows:

$$f(y) = C_0 |y|^{\frac{5\alpha}{1+\alpha}} (1 + o(1)) \quad \text{as} \quad y \to -\infty \quad \big(\tfrac{5\alpha}{1+\alpha} = \tfrac{\alpha}{\beta}\big), \tag{5.11}$$

where $C_0 > 0$ is an arbitrary constant by scaling (2.21). It is key to derive the whole 4D bundle of solutions satisfying (5.11). This is done by the linearization as $y \to -\infty$:

$$\begin{aligned} &f(y) = f_0(y) + Y(y), \quad \text{where} \quad f_0(y) = C_0(-y)^{\frac{\alpha}{\beta}} \\ \Longrightarrow \quad &-C_0((-y)^{\frac{\alpha}{\beta}} Y)^{(5)} + \beta Y'(-y) + \alpha Y + \tfrac{1}{2}\, (f_0^2(y))^{(5)} + ... = 0. \end{aligned} \tag{5.12}$$

By WKBJ-type asymptotic techniques in ODE theory, solutions of (5.12) have a standard exponential form with the characteristic equation:

$$Y(y) \sim \mathrm{e}^{a(-y)^\gamma}, \;\; \gamma = 1 + \tfrac{1}{4}\big(1 - \tfrac{\alpha}{\beta}\big) > 1 \quad \Longrightarrow \quad C_0 (\gamma a)^4 = \beta, \tag{5.13}$$

which has three roots with $\mathrm{Re}\, a_k \le 0$. Hence, we conclude that: as $y \to -\infty$,

(5.14) the bundle (5.11) is four-dimensional.

The behaviour (5.15) with the bundle (5.14) gives the desired asymptotics: by (5.11), we have the gradient blow-up behaviour at a single point: for any fixed $x < 0$, as $t \to 0^-$, where $y = x/(-t)^\beta \to -\infty$, uniformly on compact subsets,

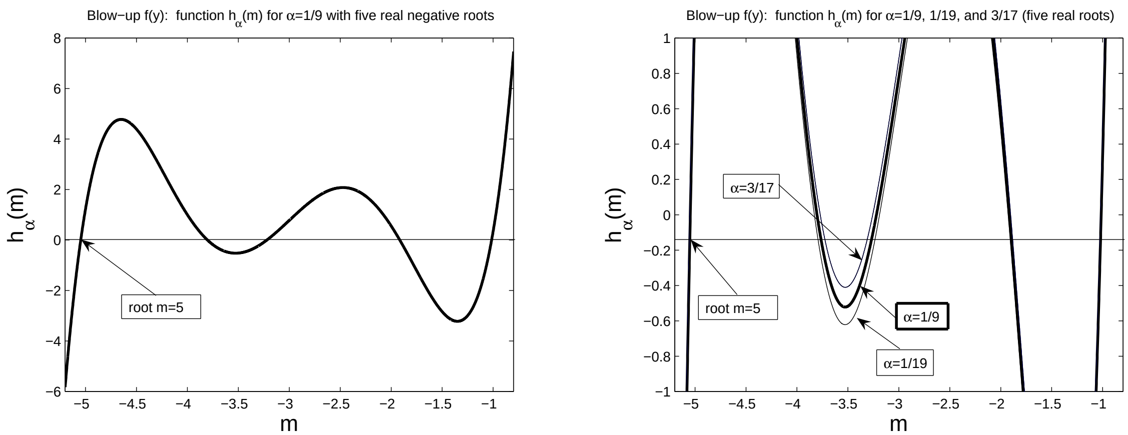
$$u_-(x,t) = (-t)^\alpha f(y) = (-t)^\alpha C_0 \big|\tfrac{x}{(-t)^\beta}\big|^{\frac{\alpha}{\beta}} (1 + o(1)) \to C_0 |x|^{\frac{5\alpha}{1+\alpha}}, \tag{5.15}$$

Let us explain some other crucial properties of the phase space, now meaning "bad bundles" of orbits. First, these are the fast growing solutions according to the explicit solution

$$f_*(y) = -\tfrac{y^5}{15120} > 0 \quad \text{for} \quad y \ll -1. \tag{5.16}$$

Analogously to (5.12), we compute the whole bundle about (5.16):

$$f(y) = f_*(y) + Y(y) \quad \Longrightarrow \quad \tfrac{1}{15120}\, (y^5 Y)^{(5)} - \beta Y' y + \alpha Y + ... = 0. \tag{5.17}$$



(a) $\alpha = \frac{1}{9}$ (b) various α

FIGURE 9. The polynomial $h_\alpha(m)$ in (5.18) for various $\alpha \in \big(0, \frac{1}{4}\big)$: five negative roots.

This Euler's equation has the following solutions with the characteristic polynomial:

$$Y(y) = y^m \quad \Longrightarrow \quad h_\alpha(m) \equiv \tfrac{(m+1)(m+2)(m+3)(m+4)(m+5)}{15120} - \beta m + \alpha = 0. \tag{5.18}$$

One root $m = -5$ is obvious that gives the solution (5.16). It turns our that this algebraic equation has precisely five negative real roots for α from the range (5.4), as Figure 9 shows. Actually, (b) explains that the graphs are rather slightly dependent on α. Thus:

$$\text{the bundle about (5.16) is five-dimensional.} \tag{5.19}$$

Second, there exists a bundle of positive solutions vanishing at some finite $y \to y_0^+ < 0$ with the behaviour (this bundle occurs from both sides, as $y \to y_0^\pm$ to be also used)

$$f_1(y) = A\sqrt{|y-y_0|}\,(1+o(1)), \quad A > 0, \tag{5.20}$$

is 4D, which also can be shown by linearization about (5.20). Indeed, the linearized operator contains the leading term

$$-A^2\big(\sqrt{|y-y_0|}\;Y\big)^{(5)} + ... = 0 \quad \Longrightarrow \quad Y(y) \sim |y-y_0|^{\frac{3}{2}}, \;\; |y-y_0|^{\frac{5}{2}}, \;\; |y-y_0|^{\frac{7}{2}}, \tag{5.21}$$

which together with the parameter $y_0 < 0$ yields

$$\text{the bundle about (5.20) is four-dimensional.} \tag{5.22}$$

Thus, (5.10), (5.14), (5.19), and (5.22) prescribe key aspects of the 5D phase space we are dealing with. To get a global orbit $\{f(y),\, y \in \mathbb{R}_-\}$ as a connection of the proper bundles (5.10) and (5.14), it is natural to follow the strategy of "shooting from below" by avoiding the bundle (5.20), (5.22), i.e., using the parameters $f_{1,3}$ in (5.10), to obtain

$$y_0 = -\infty. \tag{5.23}$$

It is not difficult to see that this profile $f(y)$ will belong to the bundle (5.14). The proof of such a 2D shooting strategy can be done by standard arguments. By scaling (2.21), we

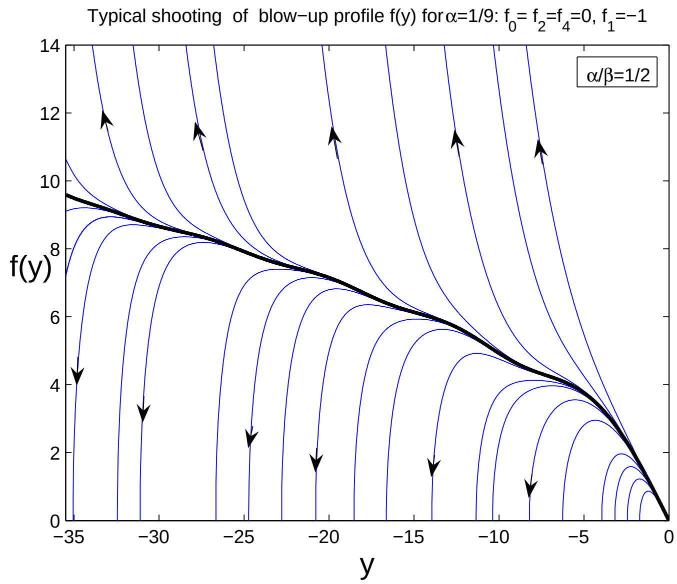


FIGURE 10. The shooting strategy of a blow-up similarity profile $f(y)$ for $\alpha = \frac{1}{9}$, with data $f(0) = f''(0) = f^{(4)} = 0$ and $f'(0) = 1$; the shooting parameter is $f'''(0) = 0.0718040128557...$.

always can reduce the problem to a 1D shooting:

$$f_1 = f'(0) = -1 \quad \text{and} \quad f_3 = f'''(0) \ \text{ is a parameter.} \tag{5.24}$$

By the above asymptotic analysis of the 5D phase space, it follows that:

(i) for $f_3 \ll -1$ the orbit belongs to the bundle about (5.16), and

(ii) for $f_3 \gg 1$, the orbit vanishes in finite y_0 along (5.20).

Hence, by continuous dependence, we obtain a solution $f(y)$ by the min-max principle (plus some usual technical details that can be omitted). Before stating the result we can prove, in Figure 10 we present how we are going to justify existence of a proper blow-up shock profile $f(y)$. Thus, we fix the above speculations as:

Proposition 5.1. (i) *In the range* (*5.4*), *the problem* (*5.2*) *admits a shock wave profile* $f(y)$*; and* (ii) *this* $f(y)$ *is unique up to scaling* (*2.21*) *and is positive for* $y < 0$.

Two results (i) and (ii) are combined for convenience, and now we declare that (ii) is an open problem. In [17], for the NDE–3 (1.28), the phase space is 3D and the proof is available.

In fact, this is a rather typical result for higher-order dynamical systems. E.g., we refer to a similar and not less complicated study of a 4th-order ODE [21], where existence and uniqueness of a positive solution of the radial bi-harmonic equation with source:

$$\Delta_r^2 u = u^p \quad \text{for} \quad r = |x| > 0, \quad u(0) = 1, \quad u'(0) = u'''(0) = 0, \quad u(\infty) = 0, \tag{5.25}$$

was proved in the supercritical Sobolev range $p > p_{\rm Sob} = \frac{N+4}{N-4}$, $N > 4$. Here, analogously, there exists a single shooting parameter being the second derivative at the origin $u_2 =$

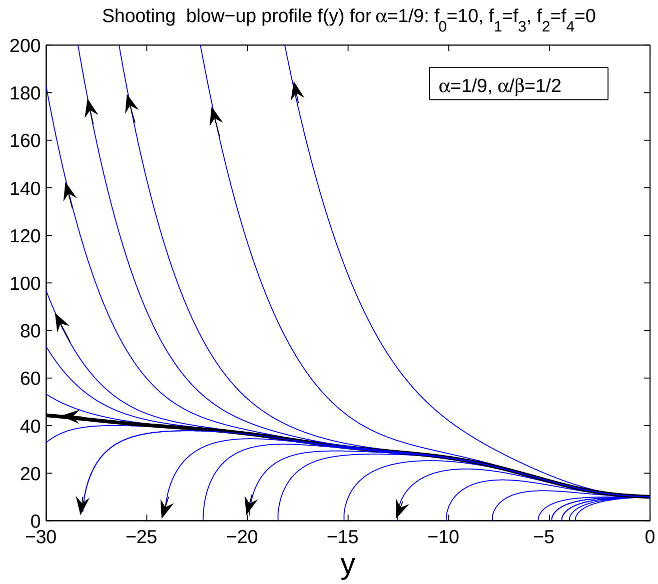


FIGURE 11. Shooting the blow-up profile $f(y)$ for $\alpha = \frac{1}{9}$; here $f(0) = 10$.

$u''(0)$; the value $u_0 = u(0) = 1$ is fixed by a scaling symmetry. Proving uniqueness of such a solution in [21] is not easy and lead to essential technicalities, which the attentive Reader can consult in case of necessity. Fortunately, we are not interested in any uniqueness of such kind. Instead of the global behaviour such as (5.16), the equation (5.25) admits the blow-up one governed by the principal operator $u^{(4)} + ... = u^p \quad (u \to +\infty)$. The solutions vanishing at finite point otherwise can be treated as in the family **(I)**.

We next use more advanced and enhanced numerical methods towards existence (and uniqueness-positivity, see (ii)) of $f(y)$. Figure 11 shows the shooting from $y = 0^-$ for

$$\alpha = \tfrac{1}{9} \quad \Longrightarrow \quad \tfrac{\alpha}{\beta} = \tfrac{1}{2}. \tag{5.26}$$

This again illustrates the actual strategy in proving Proposition 5.1. Note that here, as an illustration of another important evolution phenomenon, we solve the problem with

$$f(0) = f_0 = 10 \quad \Longrightarrow \quad [u_-(0,t)] = 2f_0(-t)^\alpha \to 0 \quad \text{as} \quad t \to 0^-, \tag{5.27}$$

so that this similarity solution describes *collapse* of a shock wave.

Next, Figure 12 shows truly blow-up profiles, with $f(0) = 0$, constructed by a different method (via the solver `bvp4c`) for convenient values $\alpha = \frac{1}{9}, \frac{1}{19}$, and $\frac{3}{17}$. Note the clear oscillatory behaviour of such patterns that is induces by complex roots of the characteristic equation (5.13).

STATIONARY SOLUTIONS WITH A "WEAK SHOCK". The ODE in (5.2) and hence the PDE (1.5) admit a number of simple continuous stationary solutions. E.g., consider

$$\alpha = \tfrac{1}{9}, \ \tfrac{\alpha}{\beta} = \tfrac{1}{2}: \quad \hat f(y) = \sqrt{|y|}\, \text{sign}\, y \quad \text{and} \quad \hat u(x,t) \equiv \sqrt{|x|}\, \text{sign}\, x. \tag{5.28}$$

We will show that such weak shocks also lead to nonuniqueness.

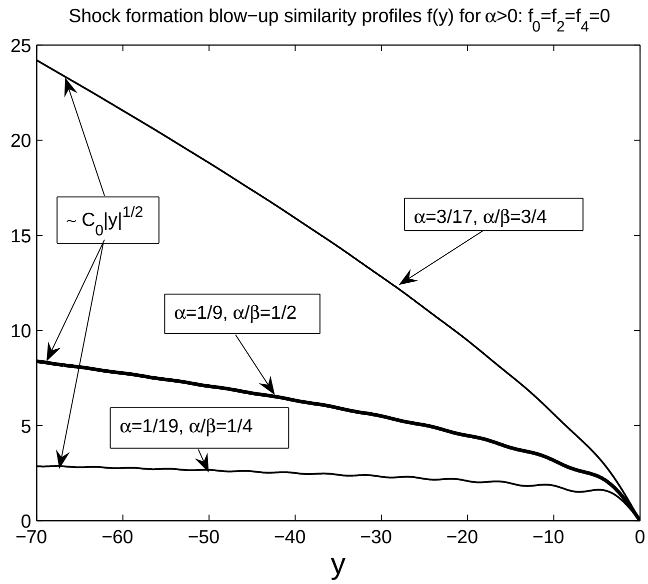


FIGURE 12. The odd blow-up similarity profiles $f(y)$ in $\mathbb{R}_-$ with $\alpha = \frac{1}{9}$, $\frac{1}{19}$, and $\frac{3}{17}$.

5.3. **On nonuniqueness of similarity extensions beyond blow-up.** A discontinuous shock wave extension of blow-up solutions (5.1), (5.2) is done by using the global ones (5.5), (5.6). Actually, this leads to watching a whole 5D family of solutions parameterized by their Cauchy values at the origin:

$$F(0) = F_0 > 0, \;\; F'(0) = F_1 < 0, \;\; F''(0) = F_2, \;\; F'''(0) = F_3, \;\; F^{(4)}(0) = F_4. \tag{5.29}$$

Thus, unlike (5.10), the proper bundle in (5.29) is 5D. Note that at minus infinity, the solution must have the form

$$F(y) = C_0 |y|^{\frac{5\alpha}{1+\alpha}}(1+o(1)) \quad \text{as} \quad y \to -\infty \quad (C_0 > 0). \tag{5.30}$$

As above, the 5D phase space for the ODE in (5.6) has two stable "bad" bundles:

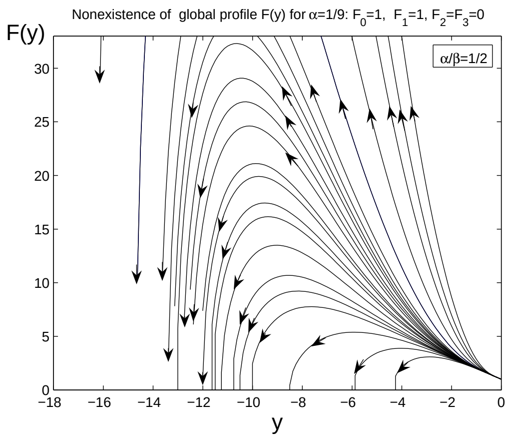
(I) Positive solutions with "singular extinction" in finite y, where $F(y) \to 0$ as $y \to y_0^+ < 0$. This is an unavoidable singularity following from the degeneracy of the equations with the principal term $FF^{(5)}$ leading to the singular potential $\sim \frac{1}{F}$. As in (5.21), this bundle is 4D, and

(II) Negative solutions with the fast growth (cf. (5.16)):

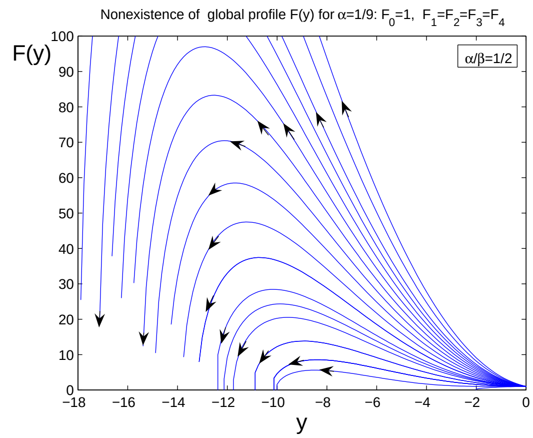
$$F_*(y) = \frac{y^5}{15120}(1+o(1)) \to -\infty \quad \text{as} \quad y \to -\infty. \tag{5.31}$$

The characteristic polynomial is the same as in (5.18), so that the bundle is 5D; cf. (5.19).

Both sets of such solutions are *open* by the standard continuous dependence of solutions of ODEs on parameters. The whole bundle of solutions satisfying (5.11) is obtained by



(a) $F_1 = -1$, $F_2 = F_3 = 0$



(b) $F_1 = F_2 = F_3 = F_4$

FIGURE 13. Examples of nonexistence of $F(y)$ via shooting from $y = 0^-$.

linearization as $y \to -\infty$ in (5.6):

$$\begin{aligned} &f(y) = F_0(y) + Y(y), \quad \text{where} \quad F_0(y) = C_0(-y)^{\frac{\alpha}{\beta}} \\ \Longrightarrow \quad &-C_0((-y)^{\frac{\alpha}{\beta}} Y)^{(5)} - \beta Y'(-y) - \alpha Y + \tfrac{1}{2} \, (F_0^2(y))^{(5)} + ... = 0. \end{aligned} \tag{5.32}$$

The WKBJ method now leads to a different characteristic equation:

$$Y(y) \sim \mathrm{e}^{a(-y)^\gamma}, \;\; \gamma = 1 + \tfrac{1}{4}\left(1 - \tfrac{\alpha}{\beta}\right) > 1 \quad \Longrightarrow \quad C_0(\gamma a)^4 = -\beta, \tag{5.33}$$

so that there exist just two complex conjugate roots with Re<0, and hence, unlike (5.14),

$$\text{the bundle (5.11) of global orbits } \{F(y)\} \text{ is three-dimensional.} \tag{5.34}$$

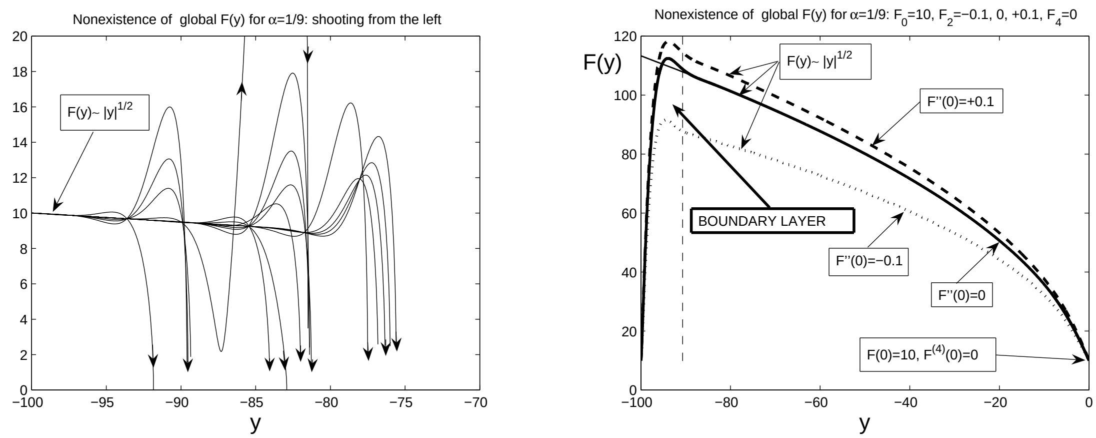
However, the geometry of the whole phase space changes dramatically in comparison with the blow-up cases, so that the standard shooting of *positive* global profiles $F(y)$ yields no encouraging results. We refer to Figure 13, which illustrate typical results of a standard shooting. In Figure 14, we show two other examples showing nonexistence of such an $F(y)$. In (a), this is done by shooting from from the left from the bundle (5.30), and in (b0 we have used the `bvp4c` solver to get convergence again to profiles with a singularity as in (5.20).

Thus, surprisingly, we arrive at a clear nonexistence of the extension similarity problem (5.6). Note that for the NDE–3 (1.28), such a nonunique profile $F(y)$ does exist [17], thus ensuring the nonuniqueness (and non-entropy) in the similarity-ODE framework for the problem. We pose two related questions:

Open Problems 5.1 (i) *To prove that the global similarity extension problem* (*5.6*) *does not have a solution* $F(y)$ *for any* $C_0 > 0$, and hence
(ii) *To describe non-self-similar formation of (nonunique) shock waves from data* (*5.3*).

NONUNIQUENESS. The justified nonuniqueness is achieved for the values of parameters

$$F(0) = F_0 < 0 \quad \text{and} \quad F'(0) = F_1 \leq 0,$$



(a) shooting from the left by `ode45` (b) by `bvp4c`

FIGURE 14. Examples of nonexistence of $F(y)$ for $\alpha = \frac{1}{9}$: by shooting from the left (a) and by iteration techniques (b).

as shown in Figure 15. The proof of existence of such profiles F is based on the same geometric arguments as that of Proposition 2.2 (with the evident change of the geometry of the phase space). These two different profiles show a nonunique way to get solutions with the initial data ($C_0 = 1$ by scaling)

$$u_0(x) = |x|^{\frac{\alpha}{\beta}} \operatorname{sign} x \quad \text{in} \quad \mathbb{R}.$$

This is another type of nonuniqueness in the Cauchy problem for (1.5), showing the nonunique way of formation of shocks from weak discontinuities, including the stationary ones as in (5.28).

5.4. **More on nonuniqueness and well-posedness of FBPs.** The non-uniqueness (5.8) in the Cauchy problem (1.5), (5.3) is: any $F(y)$ yields the self-similar continuation (5.5), with the behaviour of the jump at $x = 0$ (profiles $F(y)$ as in Figure 15)

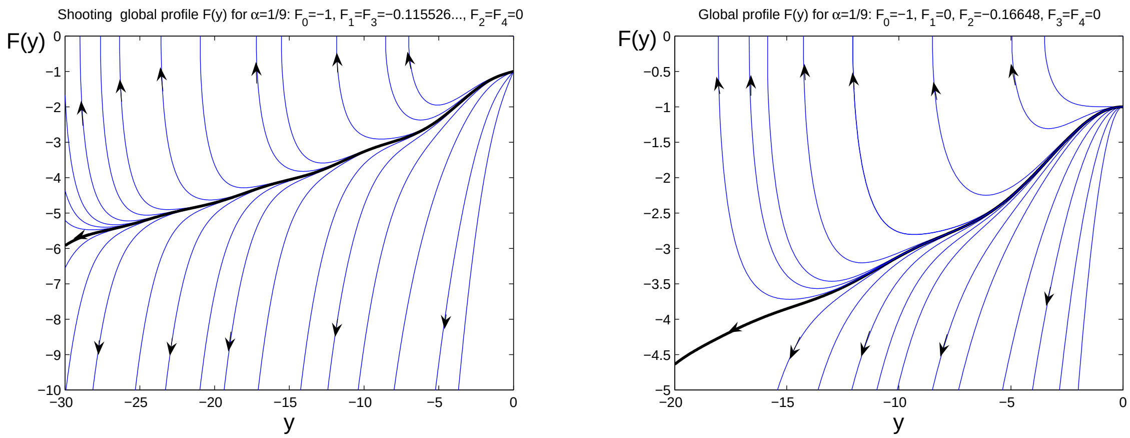
$$-[u_+(x,t)]\big|_{x=0} \equiv -\big(u_+(0^+,t) - u_+(0^-,t)\big) = 2F_0 t^\alpha < 0 \quad \text{for} \quad t > 0. \tag{5.35}$$

In the similarity ODE representation, this nonuniqueness has a pure dimensional origin associated with the dimension of the good and bad asymptotic bundles of the 5D phase spaces of both blow-up and global equations. Since these shocks are stationary, the corresponding Rankine–Hugoniot (R–H) condition on the speed λ of the shock propagation:

$$\lambda = \tfrac{[(uu_x)_{xxx}]}{[u]}\Big|_{x=0} = 0 \tag{5.36}$$

is valid by anti-symmetry. As usual, (5.36) is obtained by integration of the equation (1.28) in a small neighbourhood of the shock. The R–H condition does not assume any novelty and is a corollary of integrating the PDE about the line of discontinuity.

Moreover, the R–H condition (5.36) also indicates another origin of nonuniqueness: a *symmetry breaking*. Indeed, the solution for $t > 0$ is not obliged to be an odd function of



(a) $F'(0) = F'''(0) = -0.115526...$ (b) $F'(0) = 0,\ F''(0) = -0.16648...$

FIGURE 15. Shooting a proper solution $F(y)$ of (5.6) for $\alpha = \frac{1}{9}$ with data $F(0) = -1$, $F_4 = 0$, and $F'(0) = F'''(0) = -0.115526...$, $F_2 = 0$ (a), and $F'(0) = 0$, $F_2 = -0.16648...$, $F_3 = 0$.

x, so the self similar solution (5.5) for $x < 0$ and $x > 0$ can be defined using ten different parameters $\{F_0^\pm, ..., F_4^\pm\}$, and the only extra condition one needs is the R–H one:

$$[(FF')'''] = 0, \quad \text{i.e.,} \quad F_0^- F_4^- + 4F_1^- F_3^- + 3(F_2^-)^2 = F_0^+ F_4^+ + 4F_1^+ F_3^+ + 3(F_2^+)^2. \tag{5.37}$$

This algebraic equations with *ten* unknowns admit many other solutions rather than the obvious anti-symmetric one:

$$F_0^- = -F_0^+, \quad F_1^- = F_1^+, \quad F_2^- = -F_2^+, \quad F_3^- = F_3^+, \text{ and } F_4^- = -F_4^+.$$

Finally, we note that the uniqueness can be restored by posing specially designed conditions on moving shocks, which, overall guarantee the unique solvability of the algebraic equation in (5.37) and hence the unique continuation of the solution beyond blow-up. This construction is analytically similar to that for the NDEs–3 (1.28) in [17].

6. Shocks for an NDE obeying the Cauchy–Kovalevskaya theorem

In this short section, we touch the problem of formation of shocks for NDEs that are higher-order in time. Instead of studying the PDEs such as (cf. [16, 19])

$$u_{tt} = -(uu_x)_{xxxx}, \quad u_{ttt} = -(uu_x)_{xxxx}, \quad \text{etc.,} \tag{6.1}$$

we consider the fifth-order in time NDE (1.10), which exhibits certain simple and, at the same time, exceptional properties. Writing it for $W = (u, v, w, g, h)^T$ as

$$\begin{cases} u_t = v_x, \\ v_t = w_x, \\ w_t = g_x, \\ g_t = h_x, \\ h_t = uu_x, \end{cases} \quad \text{or} \quad W_t = AW_x, \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ u & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.2}$$

(1.10) becomes a first-order system with the characteristic equation for eigenvalues

$$\lambda^5 - u = 0,$$

so it is not hyperbolic for $u \neq 0$.

6.1. **Evolution formation of shocks.** For (1.10), the blow-up similarity solution is

$$u_-(x,t) = g(z), \quad z = x/(-t), \quad \text{where} \tag{6.3}$$

$$\begin{aligned} (gg')^{(4)} = (z^5 g')^{(4)} \equiv 120g'z + 240g''z^2 + 120g'''z^3 \\ +20g^{(4)}z^4 + g^{(5)}z^5 \quad \text{in} \quad \mathbb{R}, \quad f(\mp\infty) = \pm 1. \end{aligned} \tag{6.4}$$

Integrating (6.4) four times yields

$$gg' = z^5 g' + Az + Bz^3, \quad \text{with constants} \quad A = (g'(0))^2 > 0, \; B = \tfrac{2}{3}\, g'(0)g'''(0), \tag{6.5}$$

so that the necessary similarity profile $g(z)$ solves the first-order ODE

$$\tfrac{\mathrm{d}g}{\mathrm{d}z} = \tfrac{Az+Bz^3}{g-z^5}. \tag{6.6}$$

By the phase-plane analysis of (6.6) with $A > 0$ and $B = 0$, we easily get the following:

Proposition 6.1. *The problem* (*6.4*) *admits a solution $g(z)$ satisfying the anti-symmetry conditions* (*2.9*) *that is positive for $z < 0$, monotone decreasing, and is real analytic.*

Actually, involving the second parameter $B > 0$ yields that there exist infinitely many shock similarity profiles. The boldface profile $g(z)$ in Figure 16 (by (6.3), it gives $S_-(x)$ as $t \to 0^-$) is non-oscillatory about ± 1, with the following algebraic rate of convergence to the equilibrium as $z \to -\infty$:

$$g(z) = \begin{cases} 1 + \frac{A}{3z^5} + ... & \text{for } B = 0, \\ 1 + \frac{B}{z} + ... & \text{for } B > 0. \end{cases}$$

Note that the fundamental solutions of the corresponding linear PDE

$$u_{ttttt} = u_{xxxxx} \tag{6.7}$$

is not oscillatory as $x \to \pm\infty$. This has the form

$$b(x,t) = t^3 F(y), \quad y = x/t, \quad \text{so that } b(x,0) = ... = b_{ttt}(x,0) = 0, \; b_{tttt}(x,0) = \delta(x).$$

The linear equation (6.7) exhibits some features of finite propagation via TWs, since

$$u(x,t) = f(x - \lambda t) \quad \Longrightarrow \quad -\lambda^5 f^{(5)} = f^{(5)}, \;\; \text{i.e.,} \;\; \lambda = -1$$

and the profile $f(y)$ disappears from. This is similar to some canonical equations of mathematical physics such as

$$u_t = u_x \text{ (dispersion, } \lambda = -1) \quad \text{and} \quad u_{tt} = u_{xx} \text{ (wave equation, } \lambda = \pm 1).$$

The blow-up solution (6.3) gives in the limit $t \to 0^-$ the shock $S_-(x)$, and (1.7) holds.

Since (1.10) has the same symmetry (3.1) as (1.28), similarity solutions (6.3), with $-t \mapsto t$ and $g(z) \mapsto g(-z)$ according to (3.5), also give the rarefaction waves for $S_+(x)$ as well as other types of collapse of initial non-entropy discontinuities.

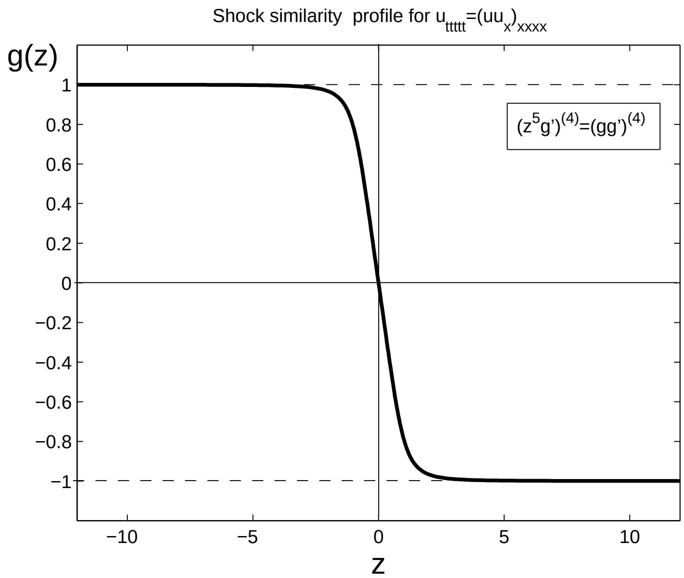


FIGURE 16. The shock similarity profile satisfying (6.4).

6.2. **Analytic δ-deformations by Cauchy–Kovalevskaya theorem.** The great advantage of the equation (1.10) is that it is in the *normal form*, so it obeys the Cauchy–Kovalevskaya theorem [45, p. 387]. Hence, for any analytic initial data $u(x,0)$, $u_t(x,0)$, $u_{tt}(x,0)$, $u_{ttt}(x,0)$, and $u_{tttt}(x,0)$, there exists a unique local in time analytic solution $u(x,t)$. Thus, (1.10) generates a local semigroup of analytic solutions, and this makes it easier to deal with smooth δ-deformations that are chosen to be analytic. This defines a special analytic δ-entropy test for shock/rarefaction waves. On the other hand, such nonlinear PDEs can admit other (say, weak) solutions that are not analytic. Actually, Proposition 6.1 shows that the shock $S_-(x)$ is a δ-entropy solution of (1.10), which is obtained by finite-time blow-up as $t \to 0^-$ from the analytic similarity solution (6.3).

6.3. **On formation of single-point shocks and extension nonuniqueness.** Similar to the analysis in Section 5, for the model (1.10) (and (6.1)), these assume studying extension similarity pairs $\{f, F\}$ induced by the easy derived analogies of the blow-up (5.2) and global (5.6) 5D dynamical systems. These are very difficult, so that checking possible nonuniqueness and non-entropy of such flows with strong and weak shocks becomes a hard open problem (some auxiliary analytic steps towards nonuniqueness are doable).

7. Nonnegative compactons of fifth-order NDEs are not robust

We begin with an easier explicit example of compactons for a third-order NDE.

7.1. **Third-order NDEs: δ-entropy compactons.** Compactons as compactly supported TW solutions of the $K(2,2)$ equation (1.22) were introduced in 1993, [39], as

$$u_c(x,t) = f_c(y), \;\; y = x+t \;\;\; \Longrightarrow \;\;\; f_c: \;\;\; f = (f^2)'' + f^2. \tag{7.1}$$

Integrating yields the following explicit compacton profile:

$$f_{\rm c}(y) = \begin{cases} \frac{4}{3}\cos^2\left(\frac{y}{4}\right) & \text{for} \quad |y| \le 2\pi, \\ 0 & \text{for} \quad |y| \ge 2\pi. \end{cases} \tag{7.2}$$

The corresponding compacton (7.1), (7.2) is a δ-entropy solution, i.e., can be constructed by smooth (and moreover analytic) approximation via strictly positive solutions, [16].

It is curious that the same compactly supported blow-up patterns occur in the combustion problem for the related reaction-diffusion parabolic equation

$$u_t = (u^2)_{xx} + u^2. \tag{7.3}$$

Then the standing-wave blow-up (as $t \to T^-$) solution of S-regime leads to the same ODE:

$$u_{\rm S}(x,t) = (T-t)^{-1} f(x) \quad \Longrightarrow \quad f = (f^2)'' + f^2. \tag{7.4}$$

This yields the *Zmitrenko–Kurdyumov blow-up localized solution*, which has been known since 1975; see more historical details in [20, § 4.2].

7.2. **Examples of C^3-smooth nonnegative compacton for higher-order NDEs.** Such an example was given in [10, p. 4734]. Following [20, p. 189], we construct this explicit solution as follows. The operator $\mathbf{F}_5(u)$ of the *quinic NDE*

$$u_t = \mathbf{F}_5(u) \equiv (u^2)_{xxxxx} + 25(u^2)_{xxx} + 144(u^2)_x, \tag{7.5}$$

is shown to preserve the 5D invariant subspace

$$W_5 = \mathcal{L}\{1, \cos x, \sin x, \cos 2x, \sin 2x\}, \tag{7.6}$$

i.e., $\mathbf{F}_5(W_5) \subseteq W_5$. Therefore, (7.5) restricted to the invariant subspace W_5 is a 5D dynamical system for the expansion coefficients of the solution

$$u(x,t) = C_1(t) + C_2(t)\cos x + C_3(t)\sin x + C_4(t)\cos 2x + C_5(t)\sin 2x.$$

Solving this yields the explicit compacton TW

$$u_{\rm c}(x,t) = f_{\rm c}(x+t), \quad \text{where} \quad f_{\rm c}(y) = \begin{cases} \frac{1}{105}\cos^4\left(\frac{y}{2}\right) & \text{for} \ |y| \le \pi, \\ 0 & \text{for} \ |y| \ge \pi. \end{cases} \tag{7.7}$$

This C_x^3 solution can be attributed to the Cauchy problem for (7.5).

The above invariant subspace analysis applies also to the 7th-order PDE

$$u_t = \mathbf{F}_7(u) \equiv D_x^7(u^2) + \beta D_x^5(u^2) + \gamma(u^2)_{xxx} + \nu(u^2)_x. \tag{7.8}$$

Here $\mathbf{F}_7$ admits W_5, if

$$\beta = 25, \ \ \gamma = 144, \ \ \text{and} \ \nu = 0.$$

Moreover [20, p. 190], the only operator $\mathbf{F}_7$ in (7.8) preserving the 7D subspace

$$W_7 = \mathcal{L}\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x\} \tag{7.9}$$

is in the following NDE–7:

$$u_t = \mathbf{F}_7(u) \equiv D_x^7(u^2) + 77 D_x^5(u^2) + 1876(u^2)_{xxx} + 14400(u^2)_x. \tag{7.10}$$

This makes it possible to reduce (7.10) on W_7 to a complicated dynamical system.

7.3. **Why nonnegative compactons for fifth-order NDEs are not robust.** As usual, by robustness of such solutions we mean that these are stable with respect to small perturbations of the parameters entering the NDE and the corresponding ODEs. In other words, NDEs admitting such non-negative "heteroclinic" orbits $0 \to 0$ are not *structurally stable* in a natural sense. This reminds the classic Andronov–Pontriagin–Peixoto theorem, where one of the four conditions for the structural stability of dynamical systems in $\mathbb{R}^2$ reads as follows [32, p. 301]:

$$\text{"(ii) there are no trajectories connecting saddle points... ."} \tag{7.11}$$

Actually, nonnegative compactons are special homoclinics of the origin, and we will show that the nature of their non-robustness is in the fact that these represent a stable-unstable manifold of the origin consisting of a single orbit. Therefore, in consistency with (7.11), the origin is indeed a saddle in $\mathbb{R}^4$.

In order to illustrate the lack of such a robustness, consider the NDE (7.5), where, on integration, we obtain the following ODE:

$$u_{\rm c}(x,t) = f_{\rm c}(x+t) \quad \Longrightarrow \quad f_{\rm c}: \quad 2f = (f^2)^{(4)} + ... , \tag{7.12}$$

where we omit the lower-order terms. Looking for the compacton profile $f \ge 0$, we set $f^2 = F$ to get

$$F^{(4)} = 2\sqrt{F} + ... \quad \text{for} \quad y > 0, \quad F'(0) = F'''(0) = 0. \tag{7.13}$$

As usual, we look for a symmetric $F(y)$ by putting two symmetry conditions at the origin.

Let $y = y_0 > 0$ be the interface point of $F(y)$. Then, looking for the expansion as $y \to y_0^-$ in the form

$$F(y) = \tfrac{1}{840^2}\,(y_0 - y)^8 + \varepsilon(y), \quad \text{with} \quad \varepsilon(y) = o((y_0-y)^8), \tag{7.14}$$

we obtain Euler's equation for the perturbation $\varepsilon(y)$,

$$\tfrac{1}{840}(y_0-y)^4 \varepsilon^{(4)} - \varepsilon = 0. \tag{7.15}$$

Hence, $\varepsilon(y) = (y_0 - y)^m$, with the characteristic equation

$$m(m-1)(m-2)(m-3) - 840 = 0 \Longrightarrow m_1 = -4, \;\; m_{2,3} = \tfrac{3 \pm {\rm i}\sqrt{111}}{2}, \quad m_4 = 7. \tag{7.16}$$

Hence, ${\rm Re}\, m_i < 8$, and, in other words, (7.15) does not admit any nontrivial solution satisfying the condition in (7.14); see further comments in [20, p. 142]. In fact, it is easy to see that (7.15) with $\varepsilon = 0$ is the unique positive smooth solution of $F^{(4)} = 2\sqrt{F}$.

$$\text{Thus: the asymptotic bundle of solutions (7.14) is 1D,} \tag{7.17}$$

where the only parameter is the position of the interface $y_0 > 0$. Obviously, as a typical property, this 1D bundle is not sufficient to satisfy (by shooting) TWO conditions at the origin in (7.13), so such TW profiles $F(y) \ge 0$ are nonexistent for almost all NDEs like that. In other words, the condition of positivity of the solution,

$$\text{to look for a nontrivial solution } F \ge 0 \text{ for the ODE in (7.13)} \tag{7.18}$$

creates a free-boundary "obstacle" problem that, in general, is inconsistent. Skipping the obstacle condition (7.18) will return such ODEs (or elliptic equation), with a special extension, into the consistent variety, as we will illustrate below.

Thus, nonnegative TW compactons are not generic (robust) solutions of $(2m+1)$th-order quadratic NDEs with $m = 2$, and also for larger m's, where (7.17) remains valid.

7.4. **Nonnegative compactons are robust for third-order NDEs only.** The third-order case $m = 1$, i.e., NDEs such as (1.22), is the only one where propagation of perturbations via nonnegative TW compactons is stable with respect to small perturbation of the parameters (and nonlinearities) of equations. Mathematically speaking, then the 1D bundle in (7.17) perfectly matches with the SINGLE symmetry condition at the origin,

$$F'' = 2\sqrt{F} + ... \quad \text{and} \quad F'(0) = 0.$$

7.5. **Compactons of changing sign are robust.** As a typical example, we consider the perturbed version (1.11) of the NDE–(1,4) (1.5). As we have mentioned, this is is written for solutions of changing sign, since nonnegative compactons do not exist in general. Looking for the TW compacton (7.12) yields the ODE

$$f = -\tfrac{1}{2}\,(|f|f)^{(4)} + \tfrac{1}{2}\,|f|f \quad \Longrightarrow \quad F^{(4)} = F - 2|F|^{-\frac{1}{2}}F \quad (F = |f|f). \tag{7.19}$$

Such ODEs with non-Lipschitz nonlinearities are known to admit countable sets of compactly supported solutions, which are studied by a combination of Lusternik–Schnirel'man and Pohozaev's fibering theory; see [18].

In Figure 17, we present the first TW compacton patterns (the boldface line) and the second one that is essentially non-monotone. These look like standard compacton profiles but careful analysis of the behaviour near the finite interface at $y = y_0$ shows that $F(y)$ changes sign infinitely many times according to the asymptotics

$$F(y) = (y_0 - y)^8 \varphi(s + s_0) + ..., \quad s = \ln(y_0 - y). \tag{7.20}$$

Here, the oscillatory component $\varphi(s)$ is a periodic solution of a certain nonlinear ODE and s_0 is an arbitrary phase shift; see [20, § 4.3] for further details. Thus, unlike (7.17),

(7.21) the asymptotic bundle of solutions (7.20) is 2D (parameters are y_0 and s_0),

and this is enough to match two symmetry boundary conditions given in (7.13). Such a robust solvability is confirmed by variational techniques that apply to rather arbitrary equations such as in (7.19) with similar singular non-Lipschitz nonlinearities.

Regardless the existence of such sufficiently smooth compacton solutions, it is worth recalling again that, for the NDE (1.11), as well as (2.31) and (4.3) both containing monotone nonlinearities, the generic behaviour, for other data, can include formation of shocks in finite time, with the local similarity mechanism as in Section 2.

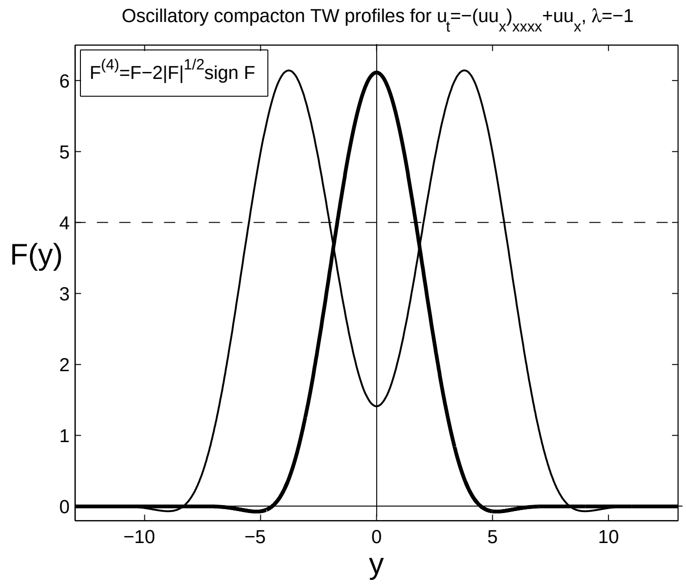


FIGURE 17. First two compacton TW profiles $F(y)$ satisfying the ODE in (7.19).

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